ROBUSTNESS PROPERTIES OF AN OPTIMAL PI CONTROLLER

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by

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CERTIFICATE

It is certified that the work contained in the thesis entitled "ROBUSTNESS PROPERTIES OF AN OPTIMAL PI CONTROLLER" has been carried out by Yashpal Singh Tomar under my supervision and that this work has not been submitted elsewhere for a degree

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ABSTRACT

A method of designing an optimal PI controller is discussed. The stability robustness and optimality robustness of the system using the optimal PI controller are discussed. The stability robustness is expressed in terms of the gain margin and phase margin, and in terms of bounds on system uncertainties which will preserve the stability of the perturbed closed loop system. The optimality robustness is expressed in terms of bounds on the perturbation in the system matrices such that the optimality of the closed loop system is preserved.

Some numerical examples are solved to illustrate the results.

TO

MY PARENTS

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CHAPTER - 1

INTRODUCTION

In this thesis the problem of designing a robust Proportional + Integral (PI) controller is considered and its stability robustness is investigated.

The servomechanism problem for systems with constant disturbances is a particular case of a more general problem which is considered by Davison [5] In his paper Davison has considered a more general class of disturbances and input for both regulator and tracking problem. The necessary and sufficient conditions were established for the existence of a robust controller for a linear time invariant, multivariable systems so that asymptotic tracking/regulation occurs independent of input disturbances and arbitrary perturbations in the plant parameters of the system It was shown that any such robust controller would consist of two devices (1) a servocompensator and (2) a stabilizing compensator The structure of the general robust servocompensator as suggested by Davison is shown in Fig. 1.1.

The servocompensator is a feed back compensator with error input, whose dynamics depends on the disturbances and reference inputs to the system. For step input and constant disturbances, the servocompensator reduces to a integral controller.

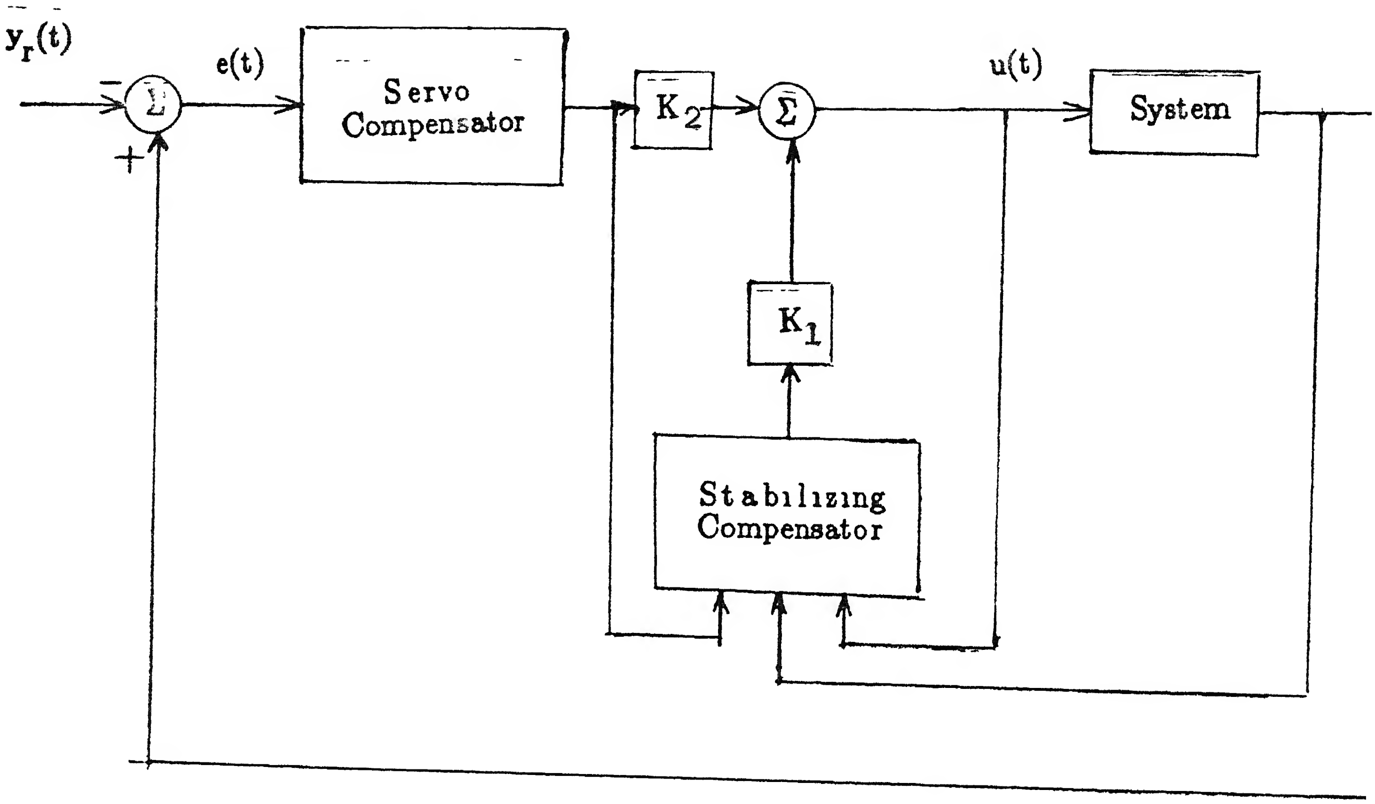

Fig 1 1 General structure and implementation of the Robust Controller

The perpose of the stabilizing compensator is to stabilize the resultant system obtained by applying the servocompensator to the plant.

The servomechanism problem has also been considered by Francis *et al.* [8] by using a geometric approach. Some other papers dealing with the special cases of the problem discussed by Davison [5] are [6], [4], [18] and [20].

The stabilizing compensator stabilizes the closed loop system after applying the servocompensator. Such a compensator is not unique and may be designed in a number of ways e.g. by pole placement using state feedback [6] or by using ouput feedback [16], [21].

In [6] state feedback approach has been used and K_1 and K_2 (in Fig. 1.1) are calculated so as to place the closed loop poles at the desired location to achieve certain performance specifications. In [18] and [21] the problem of designing dynamic compensator, using output feedback for pole placement at prespecified location, is considered. This can also be used for designing stabilizing compensator.

In [7] a compensator scheme, known as complementary controller, is suggested which can be used for the stabilizing compensator. In [24] output feedback is used for stabilization. The determination of the required controller gain is converted into the solution of an optimal control problem with quadratic performance index.

In this thesis state feedback is used for stabilization. The problem of calculating gain K_1 and K_2 (in Fig. 1.1) is posed as the linear quadratic regulator (LQR) problem. Here a quadratic cost function of the derivative of system states and the error (which in the case of a regulator problem is the output) and the derivative of control inputs is minimised. The control function such obtained is a linear combination of the states of the system and the integral of the outputs of the system is

$$u^x(t) = -K_1x(t) - K_2 \int y(t)\, dt$$

The problem of designing the optimal PI controller can be solved as the standard linear quadratic regulator design problem. Then any of the algorithm available for the given regulator design can be used by replacing system matrix by augmented system matrix and input matrix by augmented input matrix

In this thesis the designing of an optimal PI controller is converted into LQ regulator problem. Therefore the robustness properties of such PI controller can be related to the robustness properties of an LQR. Hence a survey of literature dealing with the robustness of LQ regulator is given here.

Anderson and Moore [1] have shown that SISO liner quadratic regulators have infinite gain margin, 50% gain reduction tolerance and atleast 60° phase margin. Safonov and Athans [18] have studied the robustness of LQ regulators against large dynamical time varying and nonlinear perturbation in the feedbackgains. They showed that if the control weighting matrix was diagonal the

LQ regulators would have infinite gain margin, 50% gain reduction tolerance and $\pm 60^\circ$ phase margin in each channel. This is an extension of the classical rsult due to Anderson and Moore [1] to multiinput case.

In [17] Patel *et al* have expressed the robustness property of an LQR design in terms of bounds on the perturbations in the system matrices such that the optimal closed loop system remains stable. The bounds are expressed in terms of the weighting matrices. Katayama and Sesaki [12] have obtained several robust stability results derived for linear, nonlinear, time varying and dynamical perturbations in terms of bounds of perturbations and the weighting matrices assuming that the perturbations satisfy the matching condition. In [23] it is shown that the optimal state feedback law designed for nominal system can stabilize uncertain system provided the uncertainties satisfy the matching condition and are within given bounding set. Lehtomaki *et al* [13] have derived robustness result of LQR using frequency domain criteria.

OPTIMALITY ROBUSTNESS

It may happen that the optimal control law for the nominal system, in some cases, no longer remains optimal for perturbed system In view of some inevitable parameter variations in realistic systems, it might be more desirable and practical that the optimal control law remains optimal in presence of small parameter variations (optimality robustness). The obvious significance of such an optimal control law is that the associated

optimal regulator preserves the robustness properties in the presence of parameter variations.

Hole [10] has shown that optimality is preserved under a given perturbation in the system matrices if the system is designed with a certain degree of stability. Barnett and Storey [3] have found the forms of perturbations in system matrix and input matrix which will leave the optimal (linear quadratic) control law unchanged.

Fujii and Mizushima [9] have derived necessary and sufficient conditions for a multiinput optimal control with respect to a quadratic cost to remain optimal in the presence of small parameter variations. Mori and Kokame [15] have discussed necessary and sufficient conditions for the optimality robustness of a given control law when the system matrix is expressed as the interval matrix. In [23] also the robustness of the optimallity of linear quadratic regulators is studied. In [22] it is shown that for a given optimal gain, the optimality of LQ regulators is preserved under small perturbations in system matrix and input matrix by modifying weighting matrices.

The thesis is organised as follows :

The Chapter 2 deals with the design method of optimal PI controller. In Chapter 3 the stability robustness of optimal PI controller in terms of gain margin and phase margin is discussed. Chapter 4 contains the robustness properties of optimal PI

controller in terms of the upper bound on the uncertainties which will leave the system stable. Chapter 5 deals with the design of an optimal PI controller which has optimality robustness to the perturbations in system matrix and output matrix. Chapter 6 concludes the thesis.

CHAPTER - 2

DESIGN OF THE PI CONTROLLER

2.1 INTRODUCTION

In this thesis the designing method of a robust PI controller is discussed. A number of methods are available for designing a PI controller. These methods include pole placement methods, converting the PI controller problem into an optimal output feedback problem or converting the PI controller problem into an LQR problem. In this thesis the last of the above mentioned methods, i.e., converting the PI controller problem into an LQR problem is used because of excellent stability robustness properties of an LQR.

This chapter includes, the problem statement, the necessary and sufficient conditions for the existence of a solution to the problem of designing a PI controller, and the design method of the optimal PI cntroller. One numerical example is also given to illustrate the design procedure

2.2 PROBLEM STATEMENT

We consider a linear, time invariant system described by the equations

$$\dot{x}(t) = A\,x(t) + B\,u(t) + d_i \qquad (21.a)$$

$$y(t) = C\,x(t) + d_o \qquad (2.1b)$$

$$e(t) = y(t) - y_r(t) \tag{2.1c}$$

where $x(t) \in R^n$ is the state vector,

$u(t) \in R^m$ is the input vector,

$y(t) \in R^p$ is the output vector,

$y_r(t) \in R^p$ is the reference signal vector,

d_i and d_o are constant disturbance vectors,

and

$e(t) \in R^p$ is the error signal vector.

It is required to find a controller for the system (2.1) such that the resulting controlled system is stable and the error $e(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for all $x(t_o) \in R^n$ and for constant di and do and constant reference signal.

Such a controller will be a Proportional + Integral (PI) type controller which is a particular case of the generalised controller structure as suggested by Davison [5]

2.3 NECESSARY AND SUFFICIENT CONDITIONS

The theorem given below is applicable when the disturbance and the reference signal are constant. It is a special case of the generalised theorem given by Davison [5] which deals with the existence of the controller which will cause zero steady state error in the presence of a wider class of disturbances and input

reference signals. The modified version of the theorem as suited to our problem is stated here without proof.

Theorem 2.1

A controller which solves the problem stated above can always be found if and only if the following conditions are satisfied :

(i) (A,B) is a stabilizable pair

(ii) (C,A) is a detectable pair.

(iii) The number of inputs is greater than or equal to the number of outputs i e. $m \geq p$

(iv)

$$\text{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + p$$

i.e. there is no transmission zero at the origin.

Condition (iv) means that there should be no zero at origin for single input single output systems, otherwise the pole created at the origin due to the integral comtroller will get cancelled by the zero at the origin. This will result in nonzero steady state error in the presence of constant disturbance.

2.4 DESIGN

It is required to design a controller for the system (2.1) such that the resulting closed loop system is stable and the error $e(t) \longrightarrow 0$ as $t \longrightarrow \infty$ Also the controller is to be designed by converting the original problem into LQR design problem. In

the LQR design the system states are penalised in the cost function. Hence in the following steps the error e(t) is incorporated into the systems states

Rewriting equation (2.1)

$$\dot{x}(t) = A\ x(t) + B\ u(t) + d_i \tag{21.a}$$

$$y(t) = C\ x(t) + d_o \tag{2.1b}$$

$$e(t) = y(t) - y_r(t) \tag{2.1c}$$

$$\Rightarrow \quad e(t) = C\ x(t) + d_o - y_r(t)$$

Let $\dot{n}_1(t) = e(t)$

$$\Rightarrow \quad \dot{n}_1(t) = C\ x(t) + d_o - Y_r(t) \tag{2 2}$$

Combining (2.1) and (2.2) we get

$$\begin{bmatrix} \dot{x}(t) \\ \dot{n}_1(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ n_1(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} d_i \\ d_o \end{bmatrix} - \begin{bmatrix} 0 \\ y_r \end{bmatrix}$$

or

$$\dot{\overline{x}} = \overline{A}\ \overline{x}(t) + \overline{B}\ u(t) + \overline{d} - \begin{bmatrix} 0 \\ y_r \end{bmatrix} \tag{2.3}$$

where

$$\overline{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$$

$\overline{A}$ is the augmented system matrix of dimension (n+p) x (n+p)

$$\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$\bar{B}$ is the augmented input matrix of dimension (n+p) x m

$$\bar{d} = \begin{bmatrix} d_i \\ d_o \end{bmatrix}$$

and $$\bar{x}(t) = \begin{bmatrix} x(t) \\ n_1(t) \end{bmatrix} .$$

Now taking the derivative of equation (2 3), we get

$$\ddot{\bar{x}} = \bar{A}\,\dot{\bar{x}}(t) + \bar{B}\,\dot{u}(t) \qquad (2.4)$$

Defining $\tilde{x}(t) = \dot{\bar{x}}(t)$

and $$\tilde{u}(t) = \dot{u}(t) \qquad (2.5)$$

from equation (2.4) we get

$$\dot{\tilde{x}}(t) = \bar{A}\,\tilde{x}(t) + \bar{B}\,\tilde{u}(t) \qquad (2.6)$$

Here

$$\tilde{x}(t) = \dot{\bar{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{n}_1(t) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix}$$

Thus by taking the derivative of (2.3) we get a new system description (2.6) where system state contains e(t). Also the constant disturbance terms vanished.

Now we will apply LQR design method to design a controller for system (2.6) which will make the system (2.6) stable in the sense that $\tilde{x} \longrightarrow 0$ as $t \longrightarrow \infty$. Such a controller will be a linear function of state $\tilde{x}$.

Now we take as the performance index (same as in the case of the LQR)

$$J = \int_{0}^{\infty} \left[\tilde{x}^T \bar{Q} \tilde{x} + \tilde{u}^T R \tilde{u}\right] dt \qquad (2.7)$$

where $\bar{Q}$ is symmetric and atleast positive semidefinite matrix and R is a symmetric and positive definite matrix.

The $\tilde{u}$ which will minimise J subject to (2.6) will also make the augmented system (2.6) assymptotically stable under the following conditions.

[a] Pair $[\bar{A}, \bar{B}]$ is completely stabilizable

[b] The pair $[\bar{A}, \bar{D}]$ is completely detectable where $\bar{D}\,\bar{D}^T = \bar{Q}$.

If conditions (i) and (ii) of Theorem 2 for the existence of the solution to the PI controller problem are satisfied, i.e., if (C,A,B) is stabilizable and detectable, the pair $[\bar{A}, \bar{B}]$ will be stabilizable [24]. This will satisfy condition [a].

The optimum $\tilde{u}$ is given by [1]

$$\tilde{u} = - R^{-1} \bar{B}^T \bar{P}_o \tilde{X} \qquad (2.8a)$$

$$= - \tilde{K} \tilde{x} \qquad (2.8b)$$

where $\tilde{K} = R^{-1}\, \bar{B}^T\, \bar{P}_o$ and P_o is the unique symmetric positive definite solution of the following algebraic Riccati equation

$$\bar{A}^T\, \bar{P}_o + \bar{P}_o\, \bar{A} - \bar{P}_o\, \bar{B}\, R^{-1}\, \bar{B}^T\, \bar{P}_o + \bar{Q} \;=\; 0 \qquad (2.9)$$

From equation (2.8) and using the facts

$$\tilde{u} \;=\; \dot{u}$$

and $$\tilde{x} \;=\; \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix}$$

We get

$$\dot{u}(t) \;=\; -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \qquad (2.10)$$

$$\Rightarrow \quad \dot{u}(t) \;=\; -\,K_1\, \dot{x}(t) - K_2\, e(t)$$

$$\Rightarrow \quad u(t) \;=\; -\,K_1\, x(t) - K_2 \int e(t)\, dt$$

Thus we get a PI controller where K_1 is proportional gain and K_2 is integrator gain. The design method can be summarised in the following steps.

Step 1 : Check for the controllability and observability of [C,A,B]. It should atleast be stabilizable and detectable.

Step 2 : Form augmented system matrix $\bar{A}$ and augmented input matrix $\bar{B}$.

where

$$\bar{A} \;=\; \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

Step 3 : Make proper selection of The weighting matrices R and $\bar{Q}$. $\bar{Q}$ should be selected such that the pair [$\bar{Q}^{1/2}$, $\bar{A}$] is observable.

Step 4 : Solve the algebraic Riccati equation (2.9) to get $\bar{P}_o$. Use equation (2.8) to calculate $\tilde{K}$ and partition it into [K_1 K_2] as in the equation (2.10).

2.5 EXAMPLE

$$\text{System Matrix A} = \begin{bmatrix} -1.268 & -0.045 & 1.5 & 952 \\ 1.002 & -1.96 & 8.52 & 1240 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -100 \end{bmatrix}$$

$$\text{Input Matrix B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 10 & 0 \\ 0 & 100 \end{bmatrix}$$

$$\text{Output Matrix C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

It can be seen that [A,B] is controllable and [A,C] is observable. Forming augmented system matrix $\bar{A}$ and augmented input matrix $\bar{B}$ we get

$$\bar{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} = \begin{bmatrix} -1.268 & -0.045 & 1.5 & 952 & 0 & 0 \\ 1.002 & -1.96 & 8.52 & 1240 & 0 & 0 \\ 0 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -100 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 10 & 0 \\ 0 & 100 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Taking $R = I_2$ and $\bar{Q} = I_6$.

It can be seen that $[\ \bar{A},\ \bar{Q}^{1/2}\ ]$ is observable. Using LQR design algorithm optimal gain $\tilde{K}$ is obtained

$$\tilde{K} = \begin{bmatrix} -.592869 & .45601 & .61191 & 3.06\text{x}10^{-3} & -.803 & 596 \\ .599 & 7985 & 3\ 06\text{x}10^{-2} & 4.763 & .5958 & .805 \end{bmatrix}$$

Hence proportional gain

$$K_1 = \begin{bmatrix} -.592869 & .45601 & 61191 & 3.06\text{x}10^{-3} \\ .599 & .7985 & 3.06\text{x}10^{-2} & 4.763 \end{bmatrix}$$

and Integral gain

$$K_2 = \begin{bmatrix} -803 & .596 \\ .5958 & .8025 \end{bmatrix}$$

2.6 CONCLUSION

The design method of the optimal PI controller is discussed and the necessary and sufficient conditions for the existence of PI controller are studied The design is done by converting the PI controller problem into the standard LQR problem which enables us to use the standard algorithms available for solving LQR problem for calculating the proportional gain K_1 and the integral gain K_2.

CHAPTER - 3

GAIN MARGIN AND PHASE MARGIN OF SYSTEMS WITH PI CONTROLLER

3.1 INTRODUCTION

Gain margin and phase margin are frequency domain stability robustness measures. Anderson and Moore [1] have shown that scalar input linear quadratic regulators have a gain margin of 1/2 to ∞ and a phase margin of atleast $\pm 60^\circ$. An approach similar to that used in [1] will be used in the next section to evaluate the gain margin and phase margin of scalar input PI controller. Safonov and Athans [19] have extended the result due to Anderson and Moore [1] to multiinput case. Anderson and Moore [2] have used slightly different approach to extend the result to multiinput case. This approach will be used to find the gain margin and phase margin for the multivariable case of PI controller

Finally some examples are given to verify the results

3.2 GAIN MARGIN AND PHASE MARGIN CALCULATION

For evaluating the gain margin and phase margin of the PI controller we assume that the system is free from constant disturbances because the gain margin and phase margin are defined for nominal system.

Writing the system equations (2.1) free from disturbances gives

$$\dot{x} = Ax + Bu \tag{3.1a}$$

$$y = Cx \tag{3.1b}$$

Rewriting the design steps as discussed in Chapter 2 the augmented system is obtained as

$$\dot{\tilde{x}} = \bar{A}\,\tilde{x} + \bar{B}\,\tilde{u} \tag{3.2}$$

where $\bar{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$,

$\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$,

$\tilde{x} = \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix}$,

and $\tilde{u} = \dot{u}$.

Minimising the cost function

$$J = \int_{o}^{\infty} \left\{ \tilde{x}^t\,\bar{Q}\,\tilde{x} + \tilde{u}\,R\,\tilde{u} \right\} dt$$

with respect to $\tilde{u}$ and subject to (3.2) gives the optimal control law as

$$\tilde{u} = -\tilde{K}\,\tilde{x}$$

where $\tilde{K} = R^{-1}\,\bar{B}^t\,\bar{P}_o$

$\bar{P}_o$ is the unique positive definite solution of the ARE

$$\bar{A}^T \bar{P}_o + \bar{P}_o \bar{A} + \bar{Q} - \bar{P}_o \bar{B} R^{-1} \bar{B}^t \bar{P}_o = 0.$$

Starting from the ARE it is shown in [1] that (see Appendix 1)

$$\left[I+R^{-1/2} \tilde{K} \left(-j\omega I - \bar{A}\right)^{-1} \bar{B} R^{-1/2}\right]^t \left[I+R^{1/2} \tilde{K}\left(j\omega I - \bar{A}\right)^{-1} \bar{B} R^{-1/2}\right] \geq I$$

(3.3)

<u>Scalar Case</u>

Consider a single input single output (SISO) system

$$\dot{x} = Ax + bu$$

$$y = cx$$

For SISO system equation (3.3) reduces to

$$\left[1+R^{1/2}\tilde{k} \left(-j\omega I - \bar{A}\right)^{-1} \bar{b} R^{-1/2}\right]^t \left[1+R^{1/2} \tilde{k}\left(j\omega I - \bar{A}\right)^{-1} \bar{b} R^{-1/2}\right] \geq 1$$

(3.4)

where $\bar{A} = \begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}$

and $\bar{B} = \begin{bmatrix} b \\ 0 \end{bmatrix}$

and R is a scalar.

R can be put equal to one without losing generalisation. From (3 4) we get

$$\left[1 + \tilde{k} \left(-j\omega I - \bar{A}\right)^{-1} \bar{b} \right]^2 \geq 1 \tag{3.5}$$

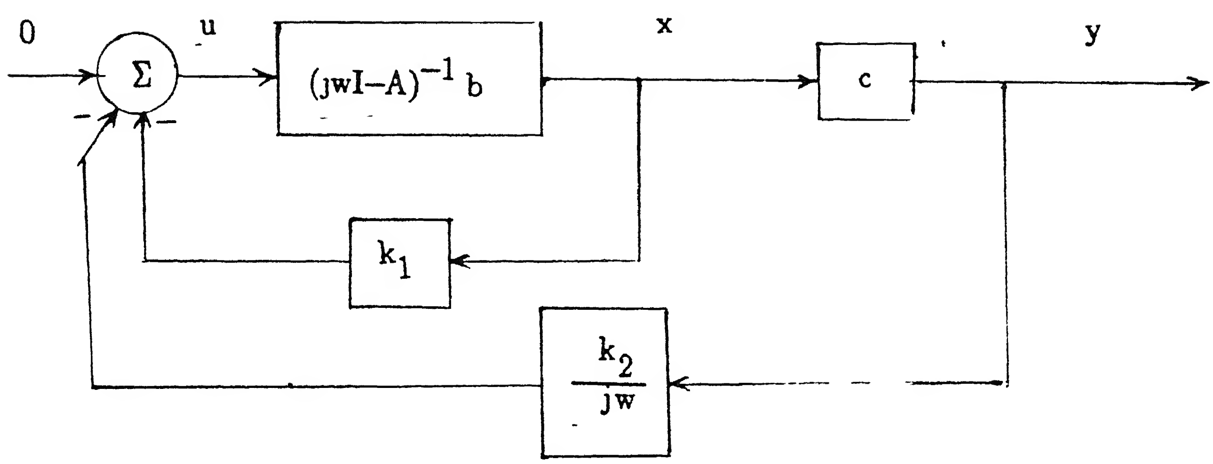

Fig 3 1(a)

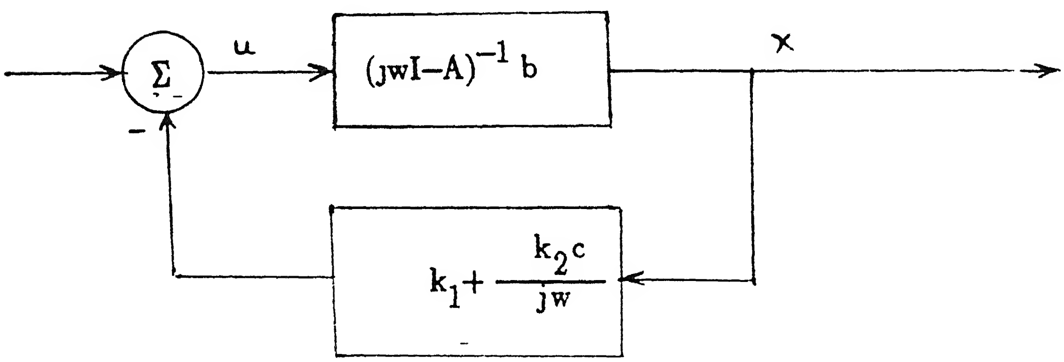

Fig 3 1(b) (Follows from Fig 3 1(a))

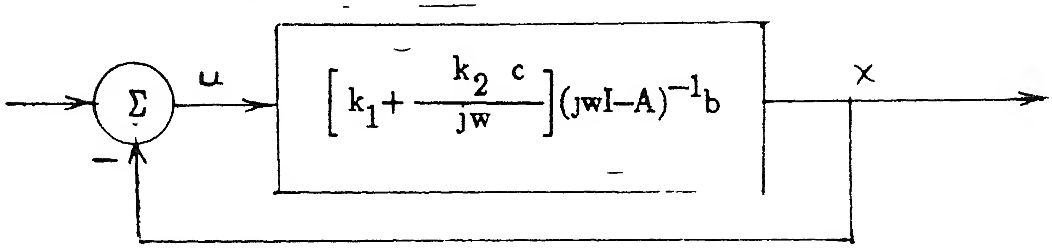

Fig 3 1(c) (Follows from Fig 3 1(b))

Block Diagram Representation of the System using PI Controller

For interpreting the equation (3 5)

replace $\bar{A}$ by $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$,

$\bar{b}$ by $\begin{bmatrix} B \\ 0 \end{bmatrix}$

and $\tilde{k}$ by [k_1 k_2].

Then from (3.5) we get

$$\left| 1 + \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} j\omega I - A & 0 \\ -c & j\omega \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} \right|^2 \geq 1$$

$$=> \left| 1 + \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1} & 0 \\ \frac{c}{j\omega} (j\omega I - A)^{-1} & \frac{1}{j\omega} \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} \right|^2 \geq 1$$

$$=> \left| 1 + \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1} b \\ \frac{1}{j\omega} c (j\omega I - A)^{-1} b \end{bmatrix} \right|^2 \geq 1$$

$$=> \left| 1 + \left(k_1 + \frac{k_2 c}{j\omega} \right) \left(j\omega I - A \right)^{-1} b \right|^2 \geq 1 \qquad (3\ 6)$$

For the interpretation of the term ($k_1 + \frac{k_2 c}{j\omega}$) see Fig. 3.1 It is obvious from the figure that the term ($k_1 + \frac{k_2 c}{j\omega}$) $(j\omega I - A)^{-1}$ b represents the open loop transfer function of the system with PI controller.

To satisfy equation (3.6) the polar plot of the open loop transfer function of the closed loop system will never enter the disc centered at $-1+j0$ point and of radius 1. Since the closed loop system is stable (ensured by the design procedure), the Nyquist plot of the system will encircle the unit disc centered at $-1+j0$ point υ times in anticlockwise direction where υ is the number of unstable poles of open loop transfer function. This ensures $\pm 60^{\circ}$ phase margin and 1/2 to ∞ gain margin. This result is the same as that obtained by Anderson and Moore [1] for linear quadratic state feedback.

Multivariable System

For evaluating the gain margin and phase margin of multivariable systems the results given in [2] are used here.

In Fig. 3.2 if the closed system is stable for $W(j\omega) = I$ and if

$$\left[I + V^{*}\left(j\omega\right)\right]\left[I + V\left(j\omega\right)\right] \geq I \quad \text{for all } \omega.$$

($[.]^{*}$ indicates complex conjugate transpose of the matrix $[.]$) then the closed loop system shown in Fig 3.2 will be stable when

$$W^{*}(j\omega) + W(j\omega) > I$$

for all ω (for the proof see Appendix 2)

Rewriting equation (3.4)

$$\left[I+R^{1/2}\tilde{K}\left(-j\omega I - \bar{A}\right)^{-1}\bar{B}\,R^{-1/2}\right]^{t}\left[I+R^{1/2}\tilde{K}\left(j\omega I - \bar{A}\right)^{-1}\bar{B}\,R^{-1/2}\right] \geq I$$

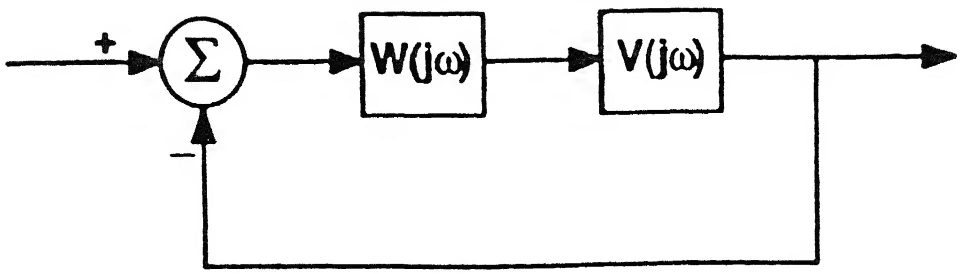

Fig 3 2 Closed Loop Systems used for Robustness Result

which can be written as

$$\left[I+\left\{R^{1/2}\tilde{K}\left(j\omega I-\bar{A}\right)^{-1}\bar{B}\,R^{-1/2}\right\}^{*}\right]\left[I+R^{1/2}\,\tilde{K}\left(j\omega I-\bar{A}\right)^{-1}\bar{B}\,R^{-1/2}\right]\geq I$$

Replacing $\bar{A}$ by $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$,

$\bar{b}$ by $\begin{bmatrix} B \\ 0 \end{bmatrix}$,

and $\tilde{K}$ by $[\ K_1\ K_2\ ]$,

we get

$$\left[I+\left\{R^{1/2}\left(K_1+\frac{K_2C}{j\omega}\right)\left(j\omega I-A\right)^{-1}B\right\}^{*}\right]$$

$$\left[I+\left\{R^{1/2}\left(K_1+\frac{K_2C}{j\omega}\right)\left(j\omega I-A\right)^{-1}B\right\}\right]\geq I \tag{3.7}$$

Redrawing the Fig 3 2, taking

$$V=R^{1/2}\left(K_1+\frac{K_2C}{j\omega}\right)\left(j\omega I-A\right)^{-1}B\,R^{-1/2}$$

and

$$W=R^{1/2}L\,R^{-1/2}$$

We get Fig. 3.3.

Using the result discussed earlier, the system shown in Fig. 3.3 will be stable when

$$R^{-1/2}L^{*}R^{1/2}+R^{1/2}L\,R^{-1/2}>I \tag{3.8}$$

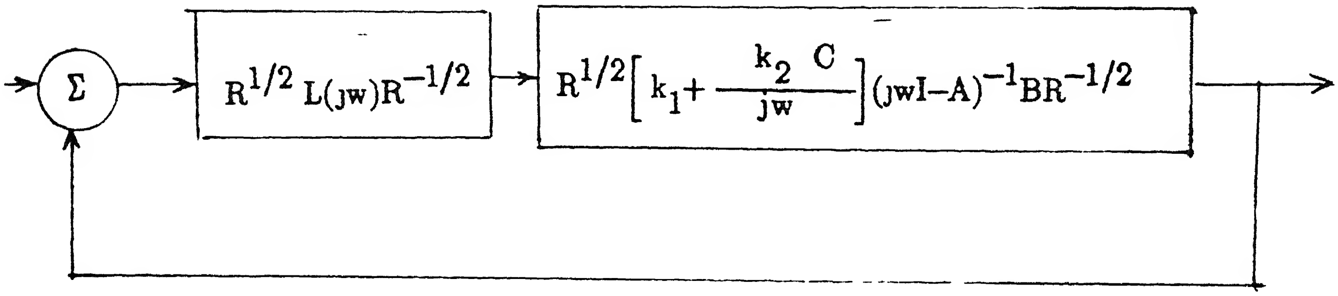

Fig 3 3

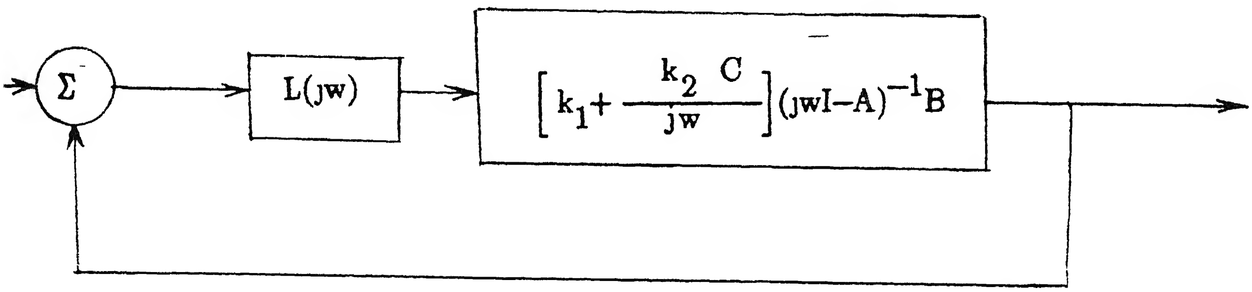

Fig 3 4 System with PI Controller with L(jw) representing perturbations

Multiplying above equation by $R^{1/2}$ from both side. We get the equivalent condition

$$L^* R + R L > R \qquad (3.9)$$

Now suppose R is diagonal and L is diag $(l_i, \ldots, l_m)$ then equation (3.9) is satisfied if and only if

$$l_i^* + l_i > 1 \qquad (i = 1, \ldots, m) \qquad (3.10)$$

If l_i is real, then (3 10) will be satisfied for $\frac{1}{2} < l_i < \infty$. This implies that there is a gain margin of $\frac{1}{2}$ to ∞ in each channel.

Also, if $l_i = e^{-j\psi}$, then (3.10) will be satisfied for $|\psi| < \frac{\pi}{3}$. This means that there is a phase margin of $\pm 60^{\circ}$ in each channel.

It can easily be seen that the two systems shown in Fig 3.3 and in Fig 3.4 have same open loop transfer function matrices. Hence the two systems have same stability properties. Therefore, the results derived above for the system shown in Fig 3.3 are also valid for the system shown in Fig. 3.4. Hence the system shown in Fig. 3.4, also have a gain margin of $\frac{1}{2}$ to ∞ in each channel and a phase margin of $\pm 60^{\circ}$ in each channel, which is a multivariable system using the PI controller designed using LQR design method.

3.3 DISCUSSION

There are a number of design methods available for PI controller. In this thesis a design method is chosen in which the proportional and integral gains are calculated by converting the

PI controller design problem into an LQR design problem for an augmented system. Now it is shown that the system with PI controller designed in this way shows a phase margin of atleast 60° and a gain margin of $\frac{1}{2}$ to ∞, same as shown by an LQR.

3.4 EXAMPLE

Here 2 examples are solved to verify the results proved in this chapter.

1.

System matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

Input vector $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and output vector $c = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

Taking $\overline{Q} = I_4$ and $R = 1$

The optimal proportional gain $k_1 = \begin{bmatrix} 11.694 & 21.516 & 34.047 \end{bmatrix}$

and the optimal proportional gain $k_2 = -3.1381$

Calculating the open loop transfer function (GH), we get

$$GH \equiv \left(k_1 + \frac{k_2 c}{s}\right)\left(sI - A\right)^{-1} b$$

$$= \frac{11.7\, s^3 + 18.16\, s^2 + 14.21\, s + 3.138}{s^4 - 4s^3 - 3s^2 - 2s}$$

GH has one unstable pole. The Nyquist plot shown in Fig. 3.5 encircles the unit disc centered at $-1+j0$ once in the counter clockwise direction as expected

2.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 4 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and $$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Taking $\overline{Q}$ $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $R = 1$,

the optimal proportional gain, we get

$$k_1 = \begin{bmatrix} 11.9 & -21.3 & -34 \end{bmatrix}$$

and the integral gain

$$k_2 = -.6$$

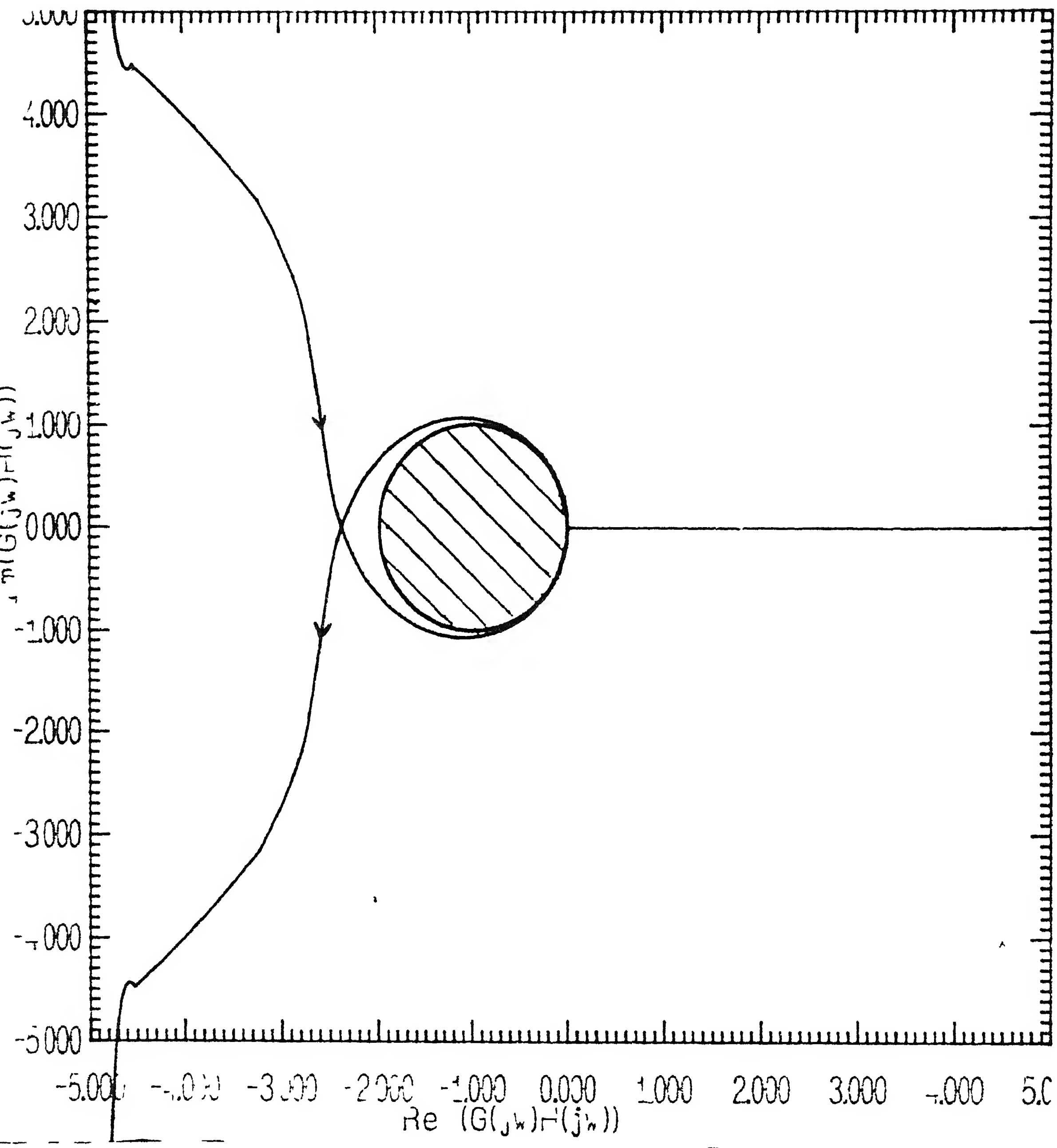

Fig 3 5 Nyquist Plot of $(11\ 7*s^3+18\ 16*s^2+14\ 21*s+3\ 14)/(s^4-4*s^3-3*s^2-2*s)$

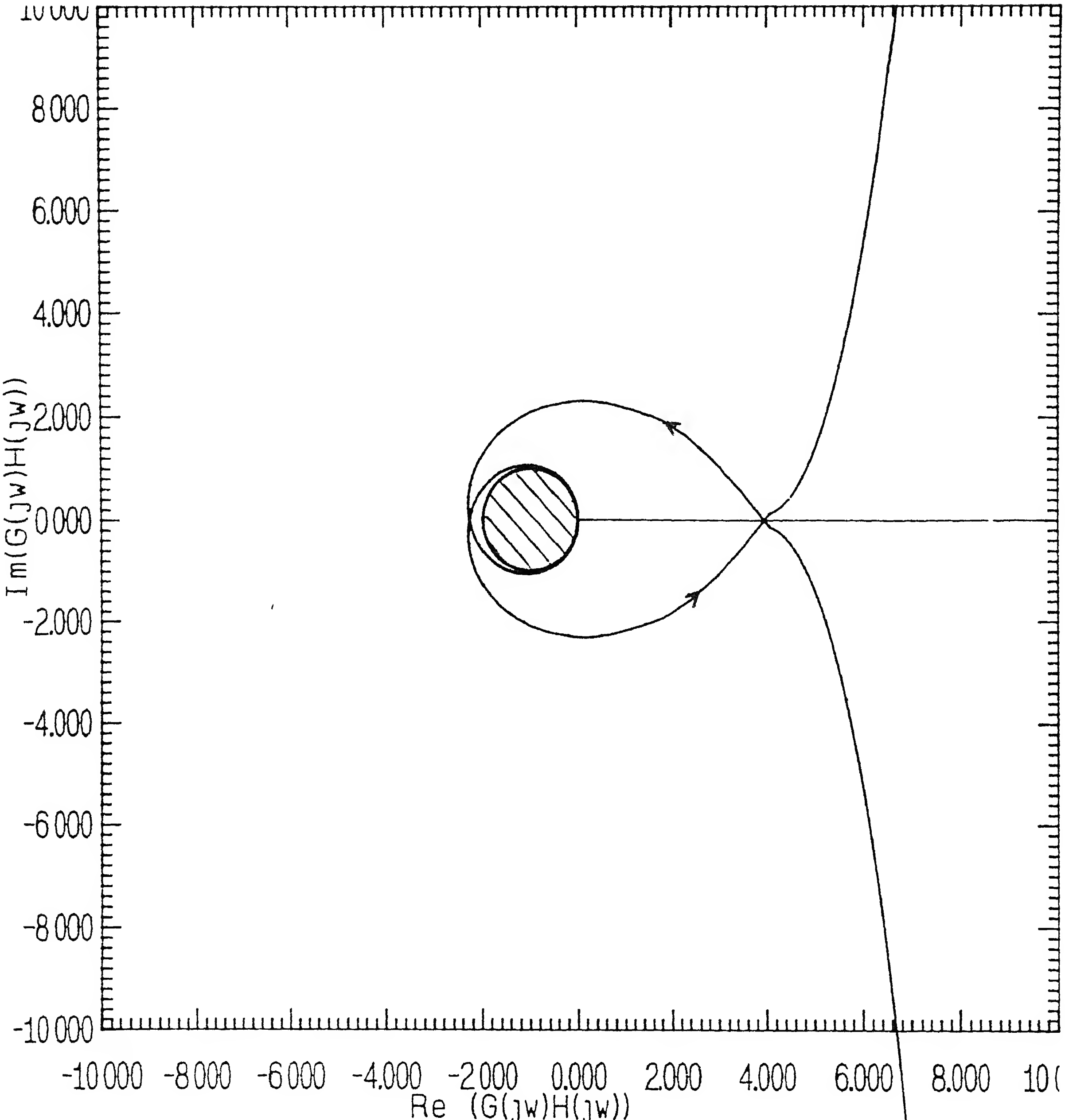

Fig.3 6 Nyqust Plot of $(11\ 9*s^3+19\ 8*s^2+10\ 5*s+3)/(s^4-4*s^3-3*s^2+2*s)$

The open loop transfer function (GH)

$$GH = \left(K_1 + \frac{K_2C_2}{s}\right)\left(sI - A\right)^{-1} b$$

$$=> \qquad GH = \frac{11.9\ s^3 + 19.8\ s^2 + 10.5\ s + 3}{s^4 - 4s^3 - 3s^2 + 2s}$$

GH has two unstable poles. The Nyquist plot (Fig. 3.6) of th system predictably encircles the unit disctwice in the count clockwise direction.

3.5 CONCLUSION

The stability robustness of the PI controller in terms gain margin and phase margin is studied. It is shown both f scalar input case and for multiinput case that the gain margin $\frac{1}{2}$ to ∞ and the phase margin is of atleast $\pm 60^\circ$. Some examp are also solved to verify the results.

CHAPTER - 4

ROBUSTNESS OF THE OPTIMAL CONTROLLER IN THE PRESENCE OF SYSTEM UNCERTAINTY

4.1 INTRODUCTION

In practice, a system is usually subject to parameter variations and modelling errors which are often very large and which do not allow for an accurate mathematical representation of the system. It is therefore necessary to investigate the stability robustness of the PI controller design in the presence of system uncertainty.

The stability robustness of an LQR in the presence of system uncertainty has been studied by Patel et al. [17]. In this chapter the robustness property of the PI controller is expressed in terms of bounds on the perturbations (modelling errors or parameter variations) in the system matrices such that the closed loop system remains stable as done in [17] for an LQR design. Relations are established between allowable perturbations and the weighting matrices in the quadratic performance index, thereby helping to select appropriate weighting matrices in the quadratic performance index to attain a robust design. Also the stability robustness of the PI controller is compared with that of an LQR design by comparing the bounds on the perturbations for stability.

Finally some numerical examples are given to illustrate the results derived in this chapter.

4.2 PROBLEM STATEMENT

Consider a linear time invariant system whose dynamics are expressed by state equations given below

$$\dot{x}(t) = A\,x(t) + B\,u(t) + f[x(t),\, u(t)] \tag{4.1a}$$

$$y(t) = C\,x(t) + g[x(t)] \tag{4.1b}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$.

A, B and C represents the system model parameters, and f and g nonlinear vector functions.

Let y_r be the constant reference signal vector then $e(t) = y(t) - y_r$ as defined in Chapter 2.

Using (4.1b), we get

$$e(t) = C\,x(t) + g\,[x(t)] - y_r$$

Following the design steps discussed in Chapter 2 we get the augmented system equation

$$\dot{\tilde{X}}(t) = \bar{A}\,\tilde{x}(t) + \bar{B}\,\tilde{u}(t) + \begin{bmatrix} \dot{f}\,[x(t),\, u(t)] \\ \\ \dot{g}\,[x(t)] \end{bmatrix} \tag{4.2}$$

where $\tilde{x}(t) = \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix}$,

$\tilde{u}(t) = \dot{u}(t)$,

$$\overline{A} = \begin{bmatrix} A & 0 \\ C & o \end{bmatrix},$$

and $$\overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

In Chapter 2, the LQR design method was used for the design of PI controller. Here a slightly modifed cost function is used to design the PI controller.

The performance index to be minimised is

$$J = \int_{o}^{\infty} e^{2\alpha t} \left[\tilde{x}^t(t)\ \overline{Q}\ \tilde{x}(t) + \tilde{u}^t(t)\ R\ \tilde{u}(t)\right] dt \tag{4.3}$$

where $\overline{Q}$ and R are selected as in Chapter 2 and α is a real positive number and is used to prescribe the degree of stability of an LQR design. The optimization of the performance index (4.3) subject to the system model

$$\tilde{x}(t) = \overline{A}\ \tilde{x}(t) + \overline{B}\ \tilde{u}(t) \tag{4.4}$$

yields the control

$$\tilde{u}(t) = - R^{-1}\ \overline{B}^t\ \overline{P}_{\alpha}(t) \tag{4.5}$$

where $\overline{P}_{\alpha} > 0$ is the solution of following algebraic Riccati equation

$$\left(\overline{A} + \alpha I\right)^t \overline{P}_{\alpha} + \overline{P}_{\alpha} \left(\overline{A} + \alpha I\right) - \overline{P}_{\alpha}\ \overline{B}\ R^{-1}\ \overline{B}^t\ \overline{P}_{\alpha} + \overline{Q} = 0 \tag{4.6}$$

The resulting closed loop system is given by

$$\dot{\tilde{X}}(t) = (\bar{A} - \bar{B}\, R^{-1}\, \bar{B}^{t}\, \bar{P}_{\alpha})\tilde{x}(t) + \begin{bmatrix} \dot{f}\ [x(t)] \\ \\ \dot{g}\ [x(t)] \end{bmatrix} \qquad (4.7)$$

and is known to be stable, if

$$\begin{bmatrix} \dot{f}\ [x(t)] \\ \dot{g}\ [x(t)] \end{bmatrix} = 0$$

The problem to be investigated is to determine the bounds on

$$\begin{bmatrix} \dot{f}\ x(t) \\ \dot{g}\ x(t) \end{bmatrix}$$

which preserve the stability of (4.7). It is the stability of augmented system which ensures zero steady state error.

For finding the bounds a theorem given by Patel et al. [17] is used here. The theorem given in [17] is applicable for LQR design. The theorem is applicable here also because PI controller design problem is converted into LQR design problem. The proof of the theorem is given in the Appendix 3.

Before stating the theorem certain definitions and notations are given here.

Euclidean norm : $||w||_E$ ($||W||_E$) denotes the Euclodean norm of vector w (matrix W).

$$||W||_E = \left(\sum |W_{ij}|^2\right)^{1/2}$$

x $\lambda(W)$ denotes maximum eigen value of W and min $\lambda(W)$ denotes nimum eigen value of W.

ectral norm : $||W||_s$ denotes the spectral norm of matrix W

$$||W||_s = \max\left[\lambda\left(WW^T\right)\right]^{1/2} = \max\left[\lambda\left(W^TW\right)\right]^{1/2}.$$

eorem 4.1 : Let $\overline{D} = \overline{Q} + \overline{P}_\alpha \; \overline{B} \; R^{-1} \; \overline{B}^t \; \overline{P}_\alpha$. Then if, the onlinear vector function

$$\begin{bmatrix} \dot{f}\; x(t) \\ \dot{g}\; x(t) \end{bmatrix}$$

atisfies the condition

$$\frac{\left\|\begin{matrix}\dot{f}(x)\\ \dot{g}(x)\end{matrix}\right\|}{||\;\tilde{x}\;||_E} \leq \overline{\mu} \equiv \frac{\min\lambda(\overline{D})}{2\max\lambda(\overline{P}_\alpha)} + \frac{\alpha\min\lambda(\overline{P}_\alpha)}{\max\lambda(\overline{P}_\alpha)} \qquad (4.8)$$

for all $\tilde{x} \in R^{n+p}$

the closed loop system (4.7) is stable.

Linear Perturbations

Now a special case of the perturbations f [x(t), u(t)] and g [x(t)] will be considered. It is assumed that f [x(t), u(t)] is linear in x(t) and u(t) and g [x(t)] is linear in x(t), and can be written as

$$f\;[x(t),\; u(t)] = E\;x(t) + F\;u(t) \qquad (4.9)$$

$$g\,[x(t)] \;=\; G\,x(t) \tag{4.10}$$

where E, F and G are nxn, nxm and pxn matrices respectively.

The system equations (4.1) now becomes

$$\dot{x}(t) \;=\; (A + E)\,x(t) + (B + F)\,u(t) \tag{4.11a}$$

$$y(t) \;=\; (C + G)\,x(t) \tag{4.11b}$$

The matrices E, F and G may represent modelling errors or parameter variations in the system model (A, B, C).

The augmented system equation (as in Section 2.4) now becomes

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A+E & 0 \\ C+G & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} B + F \\ 0 \end{bmatrix} \dot{u}(t) \tag{4.12}$$

which can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \dot{u}(t) + \begin{bmatrix} F \\ 0 \end{bmatrix} \dot{u}(t)$$

$$+ \begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} \tag{4.13}$$

or

$$\dot{\tilde{x}}(t) \;=\; \bar{A}\,\tilde{x}(t) + \bar{B}\,\tilde{u}(t) + \bar{F}\,\tilde{u}(t) + \bar{E}\,\tilde{x}(t) \tag{4.14}$$

where $\tilde{x}(t)$, $\tilde{u}(t)$, $\bar{A}$ and $\bar{B}$ are defined as in Chapter 2 and

$$\overline{F} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

$$\overline{E} = \begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix}.$$

Comparing (4.14) with (4.2), we get

$$\begin{bmatrix} \dot{f}\,[x(t),\ u(t)] \\ \\ \dot{g}\,[x(t)] \end{bmatrix} = \overline{E}\,\tilde{x}(t) + \overline{F}\,\tilde{u}(t) \qquad (4.15)$$

Using the control law (4.5), (4.15) becomes

$$\begin{bmatrix} \dot{f}\,[x(t),\ u(t)] \\ \\ \dot{g}\,[x(t)] \end{bmatrix} \equiv (\overline{E} + \overline{F}\,\tilde{K}_{\alpha})\,\tilde{x}(t) \qquad (4.16)$$

where

$$\tilde{K}_{\alpha} = -\,R^{-1}\,\overline{B}^{t}\,\overline{P}_{\alpha} \qquad (4.17)$$

Using the same control law (4.5) in system (4.14) the closed loop system becomes

$$\dot{\tilde{x}}(t) = \left[\overline{A} + \overline{E} + \left(\overline{B} + \overline{F}\right)\tilde{K}_{\alpha}\right]\tilde{x}(t) \qquad (4.18)$$

The following theorem which is derived from Theorem 4.1 gives bounds on E, F and G for the stability of closed loop system (4.18).

Theorem 4.2 : If the perturbation matrices E, F and G satisfy the condition

$$\left\|\begin{matrix} E & 0 \\ G & 0 \end{matrix}\right\|_s + \left\|\begin{matrix} F \\ 0 \end{matrix}\right\|_s \left\| \tilde{K}_\alpha \right\|_S \le \mu = \frac{\min \lambda(\bar{D})}{2\max \lambda(\bar{P}_\alpha)} + \frac{\alpha \min \lambda(\bar{P}_\alpha)}{\max \lambda(\bar{P}_\alpha)} \tag{4.19}$$

then the closed loop system (4.18) is stable.

<u>Proof</u>

From (4.16)

$$\frac{\left\|\begin{matrix} \dot{f}(X(t),\ u(t)) \\ \dot{g}(x(t)) \end{matrix}\right\|_E}{||\tilde{x}(t)||_E} = \frac{||(\bar{E} + \bar{F}\,\tilde{K}_\alpha)\,\tilde{x}(t)||_E}{||\tilde{x}(t)||_E}$$

$$\le \max_{\tilde{x}(t)} \frac{||(\bar{E} + \bar{F}\,\tilde{K}_\alpha)\,\tilde{x}(t)||_E}{||\tilde{x}(t)||_E}$$

or

$$\le ||\ \bar{E} + \bar{F}\,\tilde{K}_\alpha||_s$$

$$\le ||\bar{E}||_s + ||\bar{F}||_s\ ||\tilde{K}_\alpha||_s$$

or

$$\le \left\|\begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix}\right\|_s + \left\|\begin{bmatrix} F \\ 0 \end{bmatrix}\right\|_s \left\| \tilde{K}_\alpha \right\|_s$$

Now using Theorem 4.1 we get stability condition

$$\left\|\begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix}\right\|_s + \left\|\begin{matrix} F \\ 0 \end{matrix}\right\|_s \left\| \tilde{K}_\alpha \right\|_s \le \frac{\min \lambda(\bar{D})}{2\max \lambda(\bar{P}_\alpha)} + \frac{\alpha \min \lambda(\bar{P}_\alpha)}{\max \lambda(\bar{P}_\alpha)}$$

Thus the theorem is proved.

Theorem 4.3 : If the perturbation E and F satisfy the condition

$$\left\| \begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix} \right\|_E + \left\| \begin{matrix} F \\ 0 \end{matrix} \right\|_E \left\| \tilde{K}_\alpha \right\|_s \le \mu \equiv \frac{\min \lambda(\bar{D})}{2\max \lambda(\bar{P}_\alpha)} + \frac{\alpha \min \lambda(\bar{P}_\alpha)}{\max \lambda(\bar{P}_\alpha)} \tag{4.20}$$

then the closed loop system (4.18) is stable.

<u>Proof</u>

The result of Theorem 4.3 follows from Theorem 4.2 on noting that for any matrix W, $||W||_E \ge ||W||_s$ and therefore

$$\left\| \begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix} \right\|_E + \left\| \begin{matrix} F \\ 0 \end{matrix} \right\|_E \left\| \tilde{K}_\alpha \right\|_s \le \left\| \begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix} \right\|_s + \left\| \begin{matrix} F \\ 0 \end{matrix} \right\|_s \left\| \tilde{K}_\alpha \right\|_s$$

4.3 COMPARISON OF ROBUSTNESS OF PI CONTROLLER WITH THAT OF LQR FOR LINEAR PERTURBATIONS

Consider system represented by (4.11a)

$$\dot{x}(t) = A\,x(t) + B\,u(t) + E\,x(t) + F\,u(t)$$

Let a linear quadratic state feedback (LQSF) control law be designed for the nominal system

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

The performance index to be minimised is

$$J = \int_0^\infty e^{2\alpha t} \left[x(t)^t\, Q\, x(t) + u^t(t)\, R\, u(t) \right] dt \tag{4.21}$$

where $Q \geq 0$ and $R > 0$

Assuming that (A, B) is controllable and $(A,\ Q^{1/2})$ is observable the optimization of (4.21) yields the constant gain state feedback control

$$u(t) = -R^{-1} B^t P_\alpha x(t) \tag{4.22}$$

$$= K_\alpha x(t)$$

where $K_\alpha = -R^{-1} B^t P_\alpha$.

Where $P > 0$ is the solution of the following ARE

$$\left[A + \alpha I\right]^t P_\alpha + P_\alpha \left[A + \alpha I\right] - P_\alpha B R^{-1} B^t P_\alpha + Q = 0 \tag{4.23}$$

The resulting closed loop system will be

$$\dot{x}(t) = \left[A + E + \left[B + F\right] K_\alpha\right] x(t) \tag{4.24}$$

Patel et al [17] have found bounds on perturbation matrices E and F for the stability of the closed loop system (4.23). These bounds are given by the following theorem.

Theorem 4.4 : If the perturbation matrices E and F satisfy the condition

$$||E||_E + ||F||_E\ ||K_\alpha|| \leq \mu \equiv \frac{\min \lambda(D)}{2\max \lambda(P_\alpha)} + \frac{\alpha \min \lambda(P_\alpha)}{\max \lambda(P_\alpha)} \tag{4.25}$$

then the closed loop system (4.23) is stable. Here

$$D = Q + P_{\alpha} B R^{-1} B^t P_{\alpha}$$

For comparing the stability robustness of the PI controller with that of LQSF controller in the presence of linear perturbation it is assumed that the perturbation matrices E and F are same for both the cases. Also the input weighting matrix R is same for both the cases. The state weighting matrix Q is different from $\bar{Q}$ and they are related by

$$\bar{Q} = \begin{bmatrix} Q & t \\ t & s \end{bmatrix}$$

where t and s are chosen so as to make $\bar{Q}$ atleast n.s.d

For comparing the robustness of PI controller we will use results of Theorem 4.3 and Theorem 4.4. First of all the left hand sides of equation (4.20) and equation (4.25) are compared.

From the definition of Euclidean norm

$$\left\| \begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix} \right\|_E \geq \left\| E \right\|_E \tag{4.26}$$

(equality holds when G = 0)

and

$$\left\| \begin{bmatrix} F \\ 0 \end{bmatrix} \right\|_E = \left\| F \right\|_E \tag{4 27}$$

Now it is required to compare $||\tilde{K}_{\alpha}||_s$ with $||K_{\alpha}||_s$

$$||\tilde{K}_\alpha||_s = \max\left[\lambda\left(\tilde{K}_\alpha\ \tilde{K}_\alpha^t\right)\right]^{1/2} \tag{4.28}$$

Putting $\quad K_\alpha = -R^{-1}\,\overline{B}^t\,\overline{P}_\alpha$

and using the fact that

$$\max\left[\lambda\left(\tilde{K}_\alpha\ \tilde{K}_\alpha^t\right)\right]^{1/2} = \max\left[\lambda\left(\tilde{K}_\alpha^t\ \tilde{K}_\alpha\right)\right]^{1/2}$$

We get

$$||\tilde{K}_\alpha||_s = \max\left[\lambda\left(\overline{P}_\alpha\ \overline{B}\ R^{-1}R^{-1}\overline{B}^t\ P_\alpha\right)\right]^{1/2} \tag{4.29}$$

$$= \overline{\sigma}\left[\overline{P}_\alpha\ \overline{B}\ R^{-1}\right] \tag{4.30}$$

(where $\overline{\sigma}$ [.] represents maximum singular value of [.])

$$\Rightarrow \quad ||\tilde{K}_\alpha||_s \leq \overline{\sigma}\left[\overline{P}_\alpha\right]\overline{\sigma}\left[\overline{B}\ R^{-1}\right]$$

$$\text{or} \quad \leq \overline{\sigma}\left[\overline{P}_\alpha\right]\overline{\sigma}\left\{\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1}\right\} \quad \text{since} \quad \overline{B} \equiv \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\text{or} \quad \leq \overline{\sigma}\left[\overline{P}_\alpha\right]\overline{\sigma}\begin{bmatrix} B\ R^{-1} \\ 0 \end{bmatrix}.$$

Hence

$$||\tilde{K}_\alpha||_s \leq \overline{\sigma}\left[\overline{P}_\alpha\right]\overline{\sigma}\left[B\ R^{-1}\right] \tag{4.31}$$

as $\quad \overline{\sigma}\begin{bmatrix} B\ R^{-1} \\ 0 \end{bmatrix} = \overline{\sigma}\left[B\ R^{-1}\right].$

Similarly

$$||K_\alpha||_s = \max \left[\lambda \left(K_\alpha^t K_\alpha\right)\right]^{1/2} \tag{4.32}$$

$$\Rightarrow \quad ||K_\alpha||_s = \bar{\sigma}\left[K_\alpha^t\right]$$

Putting $K_\alpha = - R^{-1} B^t P_\alpha$ we get

$$||K_\alpha||_s = \bar{\sigma}\left[P_\alpha B R^{-1}\right]$$

$$\Rightarrow \quad ||K_\alpha||_s \le \bar{\sigma}\left[P_\alpha\right] \bar{\sigma}\left[B R^{-1}\right] \tag{4.33}$$

It is obvious from equations (4.31) and (4.33) that to compare $||K_\alpha||_s$ and $||\tilde{K}_\alpha||$, it is required to compare $\bar{\sigma}\ [\bar{P}_\alpha]$ There is no direct relation between $\bar{\sigma}\ [P_\alpha]$ and $\bar{\sigma}\ [\bar{P}_\alpha]$ since they are the solution of two different ARE's (4.6) and (4.23) respectively. In literature, bounds on the maximum eigen value (which is the same as maximum singular value as P_α and $\bar{P}_\alpha$ are symmetric and p.d) of the solution of ARE are available. These bounds give an indication about which of $\bar{\sigma}\ [P_\alpha]$ and $\bar{\sigma}\ [\bar{P}_\alpha]$ is greater

Using one such result given by Mori and Derese [14] we get

$$\bar{\sigma}\left[\bar{P}_\alpha\right] \ge \bar{\beta}_1 \equiv \frac{\mathrm{tr}\ (\bar{A} + \alpha I) + \sqrt{\mathrm{tr}\,(\bar{A} + \alpha I) + \mathrm{tr}\,(\bar{Q})\ \mathrm{tr}\ (\bar{B}\ R^{-1}\ \bar{B}^t)}}{\mathrm{tr}\,(\bar{B}\ R^{-1}\ \bar{B}^t)} \tag{4.34}$$

also

$$\bar{\sigma}\left[P_{\alpha}\right] \geq \beta_1 \equiv \frac{\text{tr}\ (A + \alpha I) + \sqrt{\text{tr}(A + \alpha I) + \text{tr}(Q)\ \text{tr}\ (B\ R^{-1}\ B^t)}}{\text{tr}(B\ R^{-1}\ B^t)} \tag{4.35}$$

Here tr [.] represents trace of matrix [.].

Comparing $\bar{\beta}_1$ and β_1 term by term

$$\text{tr}\left[\bar{A} + \alpha I\right] = \text{tr}\begin{bmatrix} A+\alpha I & 0 \\ C & \alpha I \end{bmatrix} > \text{tr}\left[A + \alpha I\right] \tag{4.36}$$

$$\text{tr}\left[\bar{Q}\right] \geq \text{tr}\left[Q\right] \tag{4.37}$$

as $\bar{Q}$ is of larger dimension than Q.

(Here it is assumed that for comparison purpose $\bar{Q}$ is chosen such that it contains Q in it i.e. $\bar{Q}$ is larger form of Q).

$$\text{tr}\left[\bar{B}\ R^{-1}\ \bar{B}^t\right] = \text{tr}\left\{\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B^t & 0 \end{bmatrix}\right\}$$

$$= \text{tr}\left\{\begin{bmatrix} B\ R^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} B^t & 0 \end{bmatrix}\right\}$$

$$= \text{tr}\left\{\begin{bmatrix} B\ R^{-1}\ B^t & 0 \\ 0 & 0 \end{bmatrix}\right.$$

$$\Rightarrow \text{tr}\left[\bar{B}\ R^{-1}\ \bar{B}^t\right] = \text{tr}\left(B\ R^{-1}\ B^t\right) \tag{4.38}$$

From inequalities (4.36), (4.37) and (4.38) it follows that

$$\bar{\beta}_1 > \beta_1 \tag{4.39}$$

Also, $$\bar{\sigma}\left[\bar{P}_\alpha\right] \geq \bar{\beta}_1 \tag{4.34}$$

and $$\sigma\left[P_\alpha\right] \geq \beta_1 . \tag{4.35}$$

Though it is not possible to say any thing with certainty about the relation between $\bar{\sigma}$ $[\bar{P}_\alpha]$ and $\bar{\sigma}$ $[P_\alpha]$ but the above three relations do give an indication that $\bar{\sigma}$ $[\bar{P}_\alpha]$ tends to be greater than $\bar{\sigma}$ $[P_\alpha]$. Hence similar kind of statement can be made about $||\tilde{K}_\alpha||_s$ and $||K_\alpha||_s$ from equations (4 31) and (4.33) that $||\tilde{K}_\alpha||_s$ tends to be greater than $||K_\alpha||_s$. This is verified by numerical examples also. Using this and equations (4.26) and (4.27) it can be stated that

$$\left|\left|\begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix}\right|\right|_E + \left|\left|\begin{bmatrix} F \\ 0 \end{bmatrix}\right|\right|_E \left|\left| \tilde{K}_\alpha \right|\right|_s > \left|\left| E \right|\right|_E + \left|\left| F \right|\right|_E \left|\left| K_\alpha \right|\right|_s$$

Thus the stability margin in the left hand side of equations (4.20) and (4.25) is smaller in case of PI controller as compared to that of LQR. This implies that PI controller is inferior to LQR as far as stability robustness to the perturbation in system matrices is concerned. It is the price that is to be paid for constant disturbance rejection and zero steady state error which are not guaranteed by an LQR.

Now we will compare right hand sides of equations (4.20) and (4.25), i e., $\bar{\mu}$ and μ.

$$\bar{\mu} \equiv \frac{\min \lambda(\bar{D})}{2\max \lambda(\bar{P}_\alpha)} + \frac{\alpha \min \lambda(\bar{P}_\alpha)}{\max \lambda(\bar{P}_\alpha)} \tag{4.20}$$

$$\bar{D} = \bar{Q} + \bar{P}_\alpha \, \bar{B} \, R^{-1} \, \bar{B}^t \, \bar{P}_\alpha$$

and

$$\mu \equiv \frac{\min \lambda(D)}{2\max \lambda(P_\alpha)} + \frac{\alpha \min \lambda(P_\alpha)}{\max \lambda(P_\alpha)} \tag{4.25}$$

In the literature bounds are available on the min. singular (or eigen) value of the solution of ARE. But these bounds depend on Q and $\bar{Q}$ and even the bounds on min $\lambda(\bar{P}_\alpha)$ and min $\lambda(P_\alpha)$ can not be compared but it is seen from numerical examples that min $\lambda(\bar{P}_\alpha)$ is smaller than min $\lambda(P_\alpha)$. Now min $\lambda(\bar{D})$ and min $\lambda(D)$ will be compared.

$\min \lambda(\bar{D}) \;=\; \underline{\sigma}\left[\bar{D}\right]$ as $\bar{D}$ is symmetric p.s.d. matrix

or $\;=\; \underline{\sigma}\left[\bar{Q} + \bar{P}_\alpha \, \bar{B} \, R^{-1} \, \bar{B}^t \, P_\alpha\right]$

or $\;\geq\; \underline{\sigma}\left[\bar{Q}\right] + \underline{\sigma}\left[\bar{P}_\alpha \, \bar{B} \, R^{-1} \, \bar{B}^t \, P_\alpha\right]$

or $\;\geq\; \underline{\sigma}\left[\bar{Q}\right] + \underline{\sigma}\left[\bar{P}_\alpha\right] \underline{\sigma}\left[\bar{B} \, R^{-1} \, \bar{B}^t\right] \underline{\sigma}\left[\bar{P}_\alpha\right]$

or $\;\geq\; \underline{\sigma}\left[\bar{Q}\right] + \underline{\sigma}\left[\bar{P}_\alpha\right] \underline{\sigma}\begin{bmatrix} B \, R^{-1} B^t & 0 \\ 0 & 0 \end{bmatrix} \underline{\sigma}\left[\bar{P}_\alpha\right]$

$$\Rightarrow \quad \min \lambda(\bar{D}) \;\geq\; \underline{\sigma}\left[\bar{Q}\right] \tag{4 40}$$

as $$\underline{\sigma}\begin{bmatrix} B\ R^{-1}\ B^{t} & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Similarly it can be seen that

$$\min \lambda(D) \geq \underline{\sigma}\left[Q\right] + \underline{\sigma}\left[P_{\alpha}\right]\underline{\sigma}\left[B\ R^{-1}\ B^{t}\right]\underline{\sigma}\left[P_{\alpha}\right] \tag{4 41}$$

From the structure of Q and $\overline{Q}$ it is obvious that $\underline{\sigma}\ [Q] \geq \underline{\sigma}[\overline{Q}]$. Though nothing can be said about min $\lambda(\overline{D})$ and min $\lambda(D)$ with certainty but inequalities (4.40) and (4 41) gives an indication tha min $\lambda(D)$ is larger than min $\lambda(\overline{D})$. In most of the cases the difference between max $\lambda(\overline{P}_{\alpha})$ and max $\lambda(P_{\alpha})$ is so large that relation between μ and $\overline{\mu}$ is mostly determined by max $\lambda(P_{\alpha})$ and max $\lambda(\overline{P}_{\alpha})$. And since max $\lambda(\overline{P}_{\alpha})$ tends to be greater than max $\lambda\ (P_{\alpha})$ (from our earlier discussion), $\overline{\mu}$ tends to be smaller than μ is most of the cases. Thus the stability margin is smaller for the PI controller than for the LQR. This is the price to be paid for constant disturbance rejection.

4.4 NUMERICAL EXAMPLE

The results of numerical examples are presented in the form of tables. A linearizedd model of a gas turbine considered by Wang and Munro [24] is given by

$$\dot{x} = \begin{bmatrix} -1.268 & -0.045 & 1.5 & 952 \\ 1.002 & -1.96 & 8.52 & 1240 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -100 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 10 & 0 \\ 0 & 100 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x$$

Table 4.1 gives the results for various values of Q and $\bar{Q}$ for the above system

Table 4.2 gives the results for various values of Q and $\bar{Q}$ for the following system

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 1]\, x$$

$$\alpha = .2 \quad \text{and} \quad R = I_2$$

LQR design	PI controller design	PI controller design
For $Q = I_4$	For $\bar{Q} = \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}$	For $\bar{Q} = I_6$
$\|\|K_\alpha\|\|_s = 3.69$	$\|\|\tilde{K}_\alpha\|\|_s = 4.27$	$\|\|\tilde{K}_\alpha\|\|_s = 5.278$
$\lambda_{min}(D) = 1$	$\lambda_{min}(\bar{D}) = .007$	$\lambda_{min}(\bar{D}) = 1$
$\lambda_{min}(P_\alpha) = .0015$	$\lambda_{min}(\bar{P}_\alpha) = .0012$	$\lambda_{min}(\bar{P}_\alpha) = .0012$
$\lambda_{max}(P_\alpha) = .226$	$\lambda_{max}(\bar{P}_\alpha) = .735$	$\lambda_{max}(\bar{P}_\alpha) = 2.228$

Table 4.1

$\alpha = .25$ and $R = 1$

LQR design	PI controller design	PI controller design
For $Q = I_3$	For $\bar{Q} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$	For $\bar{Q} = I_4$
$\lVert K_\alpha \rVert_s = 12.81$	$\lVert \tilde{K}_\alpha \rVert_s = 17.43$	$\lVert \tilde{K}_\alpha \rVert_s = 25.93$
$\lambda_{min}(D) = 1$	$\lambda_{min}(\bar{D}) = 0$	$\lambda_{min}(\bar{D}) = 1$
$\lambda_{min}(P_\alpha) = 2\ 43$	$\lambda_{min}(\bar{P}_\alpha) = 421$	$\lambda_{min}(\bar{P}_\alpha) = 1.193$
$\lambda_{max}(P_\alpha) = 100\ 23$	$\lambda_{max}(\bar{P}_\alpha) = 168.621$	$\lambda_{max}(\bar{P}_\alpha) = 324.99$
For $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	For $\bar{Q} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	For $\bar{Q} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
$\lVert K_\alpha \rVert_s = 15.09$	$\lVert \tilde{K}_\alpha \rVert_s = 20.55$	$\lVert \tilde{K}_\alpha \rVert_s = 25.06$
$\lambda_{min}(D) = 0$	$\lambda_{min}(\bar{D}) = 0$	$\lambda_{min}(\bar{D}) = 0$
$\lambda_{min}(P_\alpha) = 2.75$	$\lambda_{min}(\bar{P}_\alpha) = 719$	$\lambda_{min}(\bar{P}_\alpha) = .381$
$\lambda_{max}(P_\alpha) = 136.48$	$\lambda_{max}(\bar{P}_\alpha) = 225\ 32$	$\lambda_{max}(\bar{P}_\alpha) = 310.83$

Table 4.2

From the results it can be seen that all the examples discussed in this chapter suggests the following.

$$||K_\alpha||_s \le ||\tilde{K}_\alpha||_s$$

$$\lambda_{min}(D) \ge \lambda_{min}(\overline{D})$$

$$\lambda_{min}(P_\alpha) \ge \lambda_{min}(\overline{P}_\alpha)$$

and $$\lambda_{max}(P_\alpha) \le \lambda_{max}(\overline{P}_\alpha).$$

The above inequalities imply smaller stability margin for PI controller as compared to that of an LQR design.

4.5 CONCLUSION

In this chapter sufficient conditions have been obtained for the stability of the optimal PI controller in the presence of linear and nonlinear perturbations The robustness is expressed in terms of bounds on the perturbations in the system matrices such that the closed loop system remains stable.

A comparison has been made between the stability robustness of a PI controller and that of an LQR design Some numerical examples are given to illustrate the results given in this chapter.

CHAPTER - 5

OPTIMALITY ROBUSTNESS OF THE PI CONTROLLER

5.1 INTRODUCTION

It has been shown in Chapter 3 that the optimality conditon (3.3) is responsible for excellent robustness properties in terms of gain margin and phase margin. But these properties are for nominal systems and can not be guaranteed for perturbed system. It is desirable to have these properties for perturbed system also for which the fulfilment of optimality condition is required for the perturbed system.

Hole [10] has given a sufficient condition for an LQR design which when satisfied guarantees the satisfactiuon of the optimality condition for the perturbed system using nominally optimal control law when fixed perturbations occur in plant state matrix. In this thesis similar result is given for the PI controller.

Hole and Mahanta [11] have given a sufficient condition for an LQR design, the satifaction of which gurantees specified gain margin and phase margin for the perturbed system, using the optimal control law obtained for nominal system, in the presence of perturbation in state matrix. Similar result also exists for the PI controller and is given in this thesis.

A bound on the perturbations in the system state matrix and output matrix is also obtained which will preserve the optimality of the PI controller under perturbation in the system matrix and in the output matrix The applications of the results are demonstrated through numerical examples

5.2 PRELIMINARIES AND PROBLEM FORMULATION

Consider a linear time invariant system described by the state equations

$$\dot{x}(t) = (A + \Delta A)\, x(t) + B\, u(t) \tag{5.1a}$$

$$y(t) = (C + \Delta C)\, x(t) \tag{5.1b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$.

A, B and C are the nominal state matrix, nominal input matrix and nominal output matrix respectively, and ΔA and ΔC represents perturbations in the system state matrix and in the output matrix respectively

The corresponding nominal system is described by

$$\dot{x}(t) = A\, x(t) + B\, u(t) \tag{5.2a}$$

$$y(t) = C\, x(t) \tag{5.2b}$$

Now to design the optimal PI controller the nominal augmented system equations obtained as discussed in Chapter 2 are given below.

$$\dot{\tilde{x}}(t) = \bar{A}\, \tilde{x}(t) + \bar{B}\, \tilde{U}(t) \qquad (5.3)$$

where $\tilde{x} = \begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix}$, $e(t) = y(t) - y_r(t)$ (The reference signal vector)

and

$$\tilde{u} = \dot{u}\ .$$

Similarly perturbed augmented system is described by

$$\dot{\tilde{x}} = \bar{A}_p\, \tilde{x} + \bar{B}\, \tilde{u} \qquad (5\ 4)$$

where $\bar{A}_p = \begin{bmatrix} A+\Delta A & 0 \\ C+\Delta C & 0 \end{bmatrix}$,

$$\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

Augmented perturbed system matrix $\bar{A}_p$ and augmented nominal system matrix $\bar{A}$ are related by

$$\bar{A}_p = \bar{A} + \Delta\bar{A} \qquad (5.5)$$

where

$$\Delta\bar{A} = \begin{bmatrix} \Delta A & 0 \\ \Delta C & 0 \end{bmatrix} \qquad (5.6)$$

For designing the PI controller the following cost function is minimised with respect to $\tilde{u}$, subejct to (5.3)

$$J = \int_{o}^{\infty} e^{2\alpha t} \left[\tilde{x}^t\, \bar{Q}\, \tilde{x} + \tilde{u}^t\, R\, \tilde{u}\right] dt \qquad (5.7)$$

where α is real positive scalar,

$Q \geq 0$ and $R > 0$.

Under the assumption that the pair $[\bar{A}, \bar{B}]$ is controllable and the pair $[\bar{A}, \bar{Q}^{1/2}]$ observable, a unique optimal control is given by

$$\tilde{u} = -\bar{K}_{\alpha}\tilde{x} = -R^{-1}\bar{B}^{t}\bar{P}_{\alpha}\tilde{x} \tag{5 8}$$

P_{α} in (5.8) is obtained as a unique symmetric p.d solution of the algebraic Riccati equation given below

$$\left(\bar{A} + \alpha I\right)^{t}\bar{P}_{\alpha} + \bar{P}_{\alpha}\left(\bar{A} + \alpha I\right) - \bar{P}_{\alpha}\bar{B}R^{-1}\bar{B}^{t}\bar{P}_{\alpha} + \bar{Q} = 0 \tag{5 9}$$

Using (5.8) in nominal system (5.3), we get

$$\dot{\tilde{x}}(t) = \left(\bar{A} - \bar{B}R^{-1}\bar{B}^{t}\bar{P}_{\alpha}\right)\tilde{x}(t) \tag{5.10}$$

It is shown in Appendix 1 that the closed loop system (5.10) satisfy the optimality condition given below.

$$\left[I+R^{1/2}\bar{K}_{\alpha}\left(j\omega I - \bar{A}\right)^{-1}\bar{B}R^{-1/2}\right]^{*}\left[I+R^{1/2}\bar{K}_{\alpha}\left(j\omega I - \bar{A}\right)^{-1}\bar{B}R^{-1/2}\right] \geq I \tag{5 11}$$

In Chapter 3 it is shown that the satisfaction of (5.11) ensures a phase margin of atleast $\pm 60^{\circ}$ and a gain margin of $\frac{1}{2}$ to ∞ for nominal closed loop system (5.10) using optimal control law.

Now the problem can be stated as follows.

It is required to find a condition under which the optimality condition is satisfied by the perturbed system (5.1)

using the control law designed for the nominal system, i.e , to find a condition under which the perturbed system will satisfy the following condition.

$$\left[I+R^{1/2}\,\overline{K}_\alpha\left(j\omega I-\overline{A}_p\right)^{-1}\overline{B}\,R^{-1/2}\right]^*\left[I+R^{1/2}\,\overline{K}_\alpha\left(j\omega I-\overline{A}_p\right)^{-1}\overline{B}\,R^{-1/2}\right]\geq I \tag{5.12}$$

The satisfaction of (5.12) will ensure a phase margin of atleast 60° and a gain margin of $\frac{1}{2}$ to ∞ for the perturbed system using control law designed for nominal system.

5.3 OPTIMALITY ROBUSTNESS RESULTS

The results are given in the form of the following theorems.

Theorem 5.1

Assuming that the pair $[\overline{A}_p, B]$ is contiollable and the pair $[\overline{A}_p, \overline{Q}^{1/2}]$ is observable, the control law (5.8) will satisfy the condition (5.12), if

$$F_\alpha = \left(\overline{A}_p - \overline{B}\,\overline{K}_\alpha\right)^T\overline{P}_\alpha + \overline{P}_\alpha\left(\overline{A}_p - \overline{B}\,\overline{K}_\alpha\right) + \overline{K}_\alpha^t\,R\,\overline{K}_\alpha \tag{5.13}$$

is atleast n.s.d.

Note : It can be seen that

$$F_\alpha = \Delta\overline{A}^t\,\overline{P}_\alpha + \overline{P}_\alpha\,\Delta\overline{A} - \left(\overline{Q} + 2\alpha\overline{P}_\alpha\right). \tag{5.14}$$

Proof

The proof of the above theorem follows on the same lines as given by Hole [10] for a similar result for an LQR design.

The result given in the Theorem 5.1 is extended in the Theorem 5.2, to all perturbations in system matrix A and output matrix C of the form $\lambda\Delta A$ and $\lambda\Delta C$ where λ is a scalar such that $0 \leq \lambda \leq 1$

The perturbed system equation now can be written as

$$\dot{x}(t) = (A + \lambda \Delta A)\, x(t) + B\, u(t) \tag{5.1a}$$

$$y(t) = (C + \lambda \Delta C)\, x(t) \; . \tag{5.1b}$$

Remarks

The Theorem 5.1 is useful when the perturbation matrix ΔA is fixed. The above result have applications in the design of robust decentralized PI controller for LTI interconnected system by treating interconnection matrix as a perturbation matrix $\Delta\bar{A}$. It also has applications in the design of robustoptimal PI controllers for system with multiple operating points.

Theorem 5.2

If the condition (5.14) is satisfied for any ΔA and ΔC by the control law (5.8) then it will also be satisfied for the perturbation $\lambda\Delta A$ and $\lambda\Delta C$ by the same control law for all $0 \geq \lambda \geq 1$

Proof

The following inequality will be used for the proof

If $S \geq 0$ and $T > 0$

then $\lambda S + (1 - \lambda)T \geq 0$ for all $0 \leq \lambda \leq 1$ (5.16)

where S and T are symmetric matrices.

Satisfaction of condition (5.14) gives

$$\bar{Q} + 2\alpha\bar{P}_\alpha - \left(\Delta\bar{A}^t\ \bar{P}_\alpha + \bar{P}_\alpha\ \Delta\bar{A}\right) \geq 0$$

Also

$$\bar{Q} + 2\alpha\bar{P}_\alpha > 0$$

as $\bar{Q} \geq 0$ and $\bar{P}_\alpha > 0$

Using (5.16) we obtain

$$\lambda\left[\bar{Q} + 2\alpha\bar{P}_\alpha - \left(\Delta\bar{A}^t\ \bar{P}_\alpha + \bar{P}_\alpha\ \Delta\bar{A}\right)\right] + \left(1 - \lambda\right)\left[\bar{Q} + 2\alpha\bar{P}_\alpha\right] \geq 0$$

$$\Rightarrow \quad \lambda\left(\bar{Q} + 2\alpha\bar{P}_\alpha\right) - \left(\lambda\Delta\bar{A}^t\ \bar{P}_\alpha + \bar{P}_\alpha\ \lambda\Delta\bar{A}\right)\Big] \geq 0 \qquad (5\ 17)$$

or

$$\lambda\Delta\bar{A}^t\ P_\alpha + \bar{P}_\alpha\ \lambda\Delta\bar{A} - \left(\bar{Q} + 2\alpha P_\alpha\right) \leq 0 \qquad (5.18)$$

Proved.

Remarks

The Theorem 5.2 implies that if the optimality of a closed loop system using optimal PI controller is preserved for some perturbations ΔA and ΔC in state matrix and output matrix, then the optimality will also be preserved for perturbations $\lambda\Delta A$ and $\lambda\Delta C$, where $0 \leq \lambda \leq 1$.

It is particularly useful when only one element of either A or C is subject to perturbations or the perturbations are linear function of a parameter. In such cases, results of Theorem 5.1 and Theorem 5 2 can be combined to find the range of the parameter variation which will pıeseıve the optimality of the perturbed system using optimal PI controller designed for the nominal system.

Now a modified version of the Theorem 5.1 is given which gives a condition which when satisfeied guarantees a reduced gain margin and reduced phase margin than guaranteed by optimality condition for perturbed system using optimal PI controller designed for nominal system.

Theorem 5.3

If

$$\overline{F}_\alpha = \beta \left[\left(\overline{A}_p - \overline{B}\ \overline{K}_\alpha\right)^T \overline{P}_\alpha + \overline{P}_\alpha \left(\overline{A}_p - \overline{B}\ \overline{K}_\alpha\right)\right] + \overline{K}_\alpha^t R\ \overline{K}_\alpha \qquad (5.19)$$

(where $\beta > 1$, is a scalar) is atleast n.s.d. for some $\alpha > 0$ and $\beta > 1$, then the perturbed system (5.4) using the optimal PI controller designed for optimal system, will be robustly stable with reduced phase margin of $\cos^{-1}(1 - \frac{1}{2\beta})$, reduced gain reduction tolerance of $\frac{50}{\beta}$ % and with infinite gain margin.

Proof

The proof follows on the same lines as given by Hole and Mahanta [11] for an LQR design.

If the perturbations ΔA and ΔC are not fixed and also not of the form $\lambda\Delta A$ and $\lambda\Delta C$, $0 \leq \lambda \leq 1$, then the following theorem gives useful condition, the satifaction of which preserves the optimality of the perturbed system.

<u>**Theorem 5.4**</u>

If the perturbation matrix $\Delta\bar{A}$ satisfies the condition

$$\left|\left|\Delta\bar{A}\right|\right|_s \leq \mu_a = \frac{\lambda_{min}(\bar{Q} + 2\alpha\bar{P}_\alpha)}{2\lambda_{max}(\bar{P}_\alpha)} \tag{5.20}$$

Then the perturbed system will satisfy the optimality condition (5.12).

Proof

Starting from condition (5.14), we get

$$\bar{Q} + 2\alpha\bar{P}_\alpha - \left(\Delta\bar{A}^t\ \bar{P}_\alpha + \bar{P}_\alpha\ \Delta\bar{A}\right) \geq 0$$

or

$$\Delta\bar{A}^t\ \bar{P}_\alpha + \bar{P}_\alpha\ \Delta\bar{A} \leq \bar{Q} + 2\alpha\bar{P}_\alpha$$

$$=> \quad \lambda_{max}\left[\Delta\bar{A}^t\ \bar{P}_\alpha + \bar{P}_\alpha\ \Delta\bar{A}\right] \leq \lambda_{min}\left[\bar{Q} + 2\alpha\bar{P}_\alpha\right] \tag{5.21}$$

or

$$\left|\left|\Delta\bar{A}^t\ \bar{P}_\alpha + \bar{P}_\alpha\ \Delta\bar{A}\right|\right|_s \leq \lambda_{min}\left[\bar{Q} + 2\alpha\bar{P}_\alpha\right] \tag{5.22}$$

as $||W||_s = \lambda_{max}[\ W\ ]$ where W is a symmetric matrix

Since

$$\left\|\Delta\bar{A}^t\,\bar{P}_\alpha + \bar{P}_\alpha\,\Delta\bar{A}\right\|_s \le \left\|\Delta\bar{A}^t\,\bar{P}_\alpha\right\|_s + \left\|\bar{P}_\alpha\,\Delta\bar{A}\right\|_s$$

$$\le \left\|\Delta\bar{A}^t\right\|_s\left\|\bar{P}_\alpha\right\|_s + \left\|\bar{P}_\alpha\right\|_s\left\|\Delta\bar{A}\right\|_s ,$$

form (5.22) it follows

$$\left\|\Delta\bar{A}^t\right\|_s\left\|\bar{P}_\alpha\right\|_s + \left\|\bar{P}_\alpha\right\|_s\left\|\Delta\bar{A}\right\|_s \le \lambda_{min}\left[\bar{Q} + 2\alpha\bar{P}_\alpha\right] . \qquad (5\ 23)$$

Since

$$\left\|\bar{A}^t\right\|_s = \left\|\Delta\bar{A}\right\|_s ,$$

from (5.23), we get

$$2\left\|\Delta\bar{A}\right\|_s\left\|\bar{P}_\alpha\right\|_s \le \lambda_{min}\left[\bar{Q} + 2\alpha\bar{P}_\alpha\right]$$

$$\Rightarrow \qquad \left\|\Delta\bar{A}\right\|_s \le \frac{\lambda_{min}\,[\bar{Q} + 2\alpha\bar{P}_\alpha]}{2\,\lambda_{max}\,(\bar{P}_\alpha)}$$

Proved.

While designing a robust controller, a designer usually has information of the bounds on each element of the perturbation matrices ΔA and ΔC. The following theorems give conditions which can readily be applied to evaluate the optimality robustness of the optimal PI controller.

Theorem 5.5

If the pertubation $\Delta\bar{A}$ satisfies the condition

$$\left|\left|\Delta\bar{A}\right|\right|_E \leq \frac{\lambda_{min}\ [\bar{Q} + 2\alpha\bar{P}_\alpha]}{2\ \lambda_{max}\ (\bar{P}_\alpha)} \tag{5.24}$$

then the closed loop perturbed system using optimal PI controller designed for the nominal system will satisfy optimality condition (5.12).

Proof

The proof follows from Theorem 5.4, since

$$\left|\left|\Delta\bar{A}\right|\right|_E \geq \left|\left|\Delta\bar{A}\right|\right|_s .$$

Theorem 5.6

If ϵ_A denotes some given bounds on all the elements of the perturbation matrices ΔA and ΔC, i.e , on all the elements of $\Delta\bar{A}$ Then if

$$\epsilon_A \leq \frac{1}{2\sqrt{(n+p)n}}\ \frac{\lambda_{min}\ [\bar{Q} + 2\alpha\ \bar{P}_\alpha]}{\lambda_{max}\ (\bar{P}_\alpha)} \tag{5.25}$$

the closed loop perturbed system will satisfy the optimality condition (5.12).

Proof

From the definition of Euclidean norm it follows that

$$\left|\left|\Delta\bar{A}\right|\right|_E \leq \sqrt{(n+p)xn}\ \ \epsilon_A$$

(since only (n+p)xn elements are subject to perturbation in $\Delta\bar{A}$).

Using the above relation and the result of Theorem 5.5, the proof of Theorem 5.6 follows directly

Remark

Generally all the elements of ΔA and ΔC are not subject to variation. If l elements, where $l \leq (n+p)xn$ of $\Delta\bar{A}$ are subejct to variation then the condition (5.25) will reduce to

$$\epsilon_A \leq \frac{1}{2\sqrt{l}} \frac{\lambda_{min}[\bar{Q} + 2\alpha \bar{P}_\alpha]}{\lambda_{max}(\bar{P}_\alpha)} \qquad (5.26)$$

as in this case

$$||\Delta\bar{A}||_E \leq \sqrt{l} \ \epsilon_A.$$

It may happen that for some given $\bar{Q}$ and R there is no α to satisfy the condition (5.20). In such cases a modified condition can be used, which when satisfied guarantees a phase margin less than 60°, a gain margin of ∞ and gain reduction tolerance of less than 50%. The modified condition is given in Theorem 5 7

<u>**Theorem 5.7**</u> **:**

If the perturbation matrix $\Delta\bar{A}$ satisfies the condition

$$||\Delta\bar{A}||_s \leq \mu_{ab} = \frac{\lambda_{min}\left[\bar{Q} + 2\alpha\bar{P}_\alpha + \left(\frac{\beta-1}{\beta}\right) \bar{K}_\alpha^t \ R \ \bar{K}_\alpha\right]}{2\ \lambda_{max}(\bar{P}_\alpha)} \qquad (5.27)$$

then the perturbed system using optimal control law (5.8) will have a phase margin of atleast $\cos^{-1}(1-\frac{1}{2\beta})$, a gain margin of infinity and gain reduction tolerance of $\frac{50}{\beta}$ %.

$$||\Delta\bar{A}||_E \leq \mu_{ab} \equiv \frac{\lambda_{min}\left[\bar{Q} + 2\alpha\bar{P}_\alpha + \left(\frac{\beta-1}{\beta}\right) \bar{K}_\alpha^t R \bar{K}_\alpha\right]}{2\lambda_{max}(\bar{P}_\alpha)} \tag{5.28}$$

then the perturbed system using the optimal control law (5.8), will have a phase margin of atleast $\cos^{-1}(1 - \frac{1}{2\beta})$, a gain margin of infinity and gain reduction tolerance of $\frac{50}{\beta}$ % .

Proof

Proof follows from Theorem 5.7 on noting that

$$||\Delta\bar{A}||_E \geq ||\Delta\bar{A}||_s .$$

Theorem 5.9

If ϵ_A denotes some given bounds on all the elements of perturbation matrices ΔA and ΔC, i.e., on all the elements of $\Delta\bar{A}$ Then if

$$\epsilon_A \leq \mu_{ab} \equiv \frac{1}{2\sqrt{(n+p)n}} \frac{\lambda_{min}\left[\bar{Q} + 2\alpha\bar{P}_\alpha + \left(\frac{\beta-1}{\beta}\right) \bar{K}_\alpha^t R \bar{K}_\alpha\right]}{\lambda_{max}(\bar{P}_\alpha)} \tag{5.29}$$

the closed loop perturbed system using the optimal control law (5.8) will have a phase margin of atleast $\cos^{-1}(1 - \frac{1}{2\beta})$, a gain margin of ∞ and gain reduction tolerance of $\frac{50}{\beta}$ %.

Proof

From the definition of Euclidean norm it follows that

$$||\Delta\bar{A}||_E \leq \sqrt{(n+p)xn}\ \epsilon_A$$

(since ony (n+p)xn elements are subject to perturbations in $\Delta\bar{A}$).

Using the above relation and the result of Theorem 5.8, the proof of Theorem 5.9 follows directly.

Remarks

If l elements where $l \le (n+p)xn$ of $\Delta\bar{A}$ are subejct to variation the condition(5 29) will reduce to

$$\epsilon_A \le \mu_{ab} \equiv \frac{1}{\sqrt{l}} \quad \frac{\lambda_{min}\left[\bar{Q} + 2\alpha\bar{P}_\alpha + \left(\frac{\beta-1}{\beta}\right) \bar{K}_\alpha^t \; R \; \bar{K}_\alpha\right]}{\lambda_{max}(\bar{P}_\alpha)} \tag{5 30}$$

As $\beta \longrightarrow \infty$, the phase margin tends to zero. Hence the system will be at the verge of instability, though the gain margin remains ∞.

5.4 NUMERICAL EXAMPLE

This is an example of an aircraft considered by Winsor and Roy [25]. The longitudinal motion of the aircraft is described by the 4th order system equation. The variable state matrix is

$$A+\Delta A(M_q) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1.401\text{x}10^{-4} & M_q-.558 & -1.9513 & .0133 \\ -2.505\text{x}10^{-4} & 1 & -1\ 3239 & -\ 0238 \\ -.561 & 0 & .358 & -.0279 \end{bmatrix}$$

The nominal value of M_q is -1.48 and the input matrix B is constant.

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -5.3307 & 6.447\text{x}10^{-3} & -.2669 \\ -.16 & -1.155\text{x}10^{-2} & -.2511 \\ 0 & .106 & .0862 \end{bmatrix}$$

The output matrix C is also assumed constant

$$C = \begin{bmatrix} 1.00\times10^{-3} & 1\ 12 & .4 & .23 \\ 1.13 & 2\times10^{-3} & 5\times10^{-2} & .11 \\ .3 & 2 & .6 & .1 \end{bmatrix}$$

The augmented system will be a system of order 7. The augmented state matrix and augmented input matrix wil be given by $\overline{A}_p$ and $\overline{B}$, where

$$\overline{A}_p=\overline{A}+\Delta\overline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1\ 401\times10^{-4} & M_q-\ 558 & -1.9513 & 0133 & 0 & 0 & 0 \\ -2.505\times10^{-4} & 1 & -1.3239 & -.0238 & 0 & 0 & 0 \\ -\ 561 & 0 & .3580 & -.0279 & 0 & 0 & 0 \\ 1\times10^{-3} & 1.12 & 4 & .23 & 0 & 0 & 0 \\ 1.13 & 2\times10^{-3} & 5\times10^{-2} & .11 & 0 & 0 & 0 \\ .3 & 2 & 6 & .1 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{B} = \begin{bmatrix} 0 & 0 & 0 \\ -5\ 3307 & 6.447\times10^{-3} & -.2669 \\ -.16 & -1\ 155\times10^{-2} & -.2511 \\ 0 & .106 & .0862 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking $\overline{Q} = I_7$ and $R = I_3$, when M_q takes nominal value, i.e., when $M_q = -1.48$, $\Delta\overline{A} = [0]$ When $M_q = -4.492$

$$\Delta\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.102 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and when $M_q = -.032$

$$\Delta\bar{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.448 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For these two values of $\Delta\bar{A}$, i.e., corresponding to $M_q = -4.492$ and $M_q = -.032$, the optimality condition (5.13) is satisfied for $\alpha = 0.1$ Using the result of Theorem 5.2, we can say that the condition (5.13) will be satisfied for all $\lambda\Delta\bar{A}$, $0 \leq \lambda \leq 1$, for $\alpha = 0.1$ When $\lambda = 0$, the perturbation in system matrix will be zero corresponding to $M_q = -1.48$. When $\lambda = 1$, the perturbation is $\Delta\bar{A}$ corresponding to $M_q = -4.492$ on one extreme and $M_q = -032$ on the other extreme Therefore for all $-4.492 \leq M_q \leq -032$ the perturbed system will satisfy the optimality condition (5 13) for $\alpha = 0.1$. For increasing this range of M_q for the same degree of stability, i.e., for $\alpha = 0.1$, we will have to sacrifice in terms

of phase margin and gain reduction tolerance by taking $\beta > 1$. For $\alpha = 0.1$ and $\beta = 1.2$ the perturbed system will satisfy the condition (5.19) for all $-5.117 \leq M_q \leq .728$. Thus the phase margin reduces to $\cos^{-1}(1 - \frac{1}{2\beta}) = 54.2^\circ$ and gain reduction tolerance reduces to $\frac{50}{1.2}$ % = 41 7 %.

Now the bound on the perturbation matrix $\Delta\overline{A}$ is calculated which will preserve the optimality of the perturbed system.

From (5.20),

$$\mu_a \equiv \frac{\lambda_{max}(\overline{Q} + 2\alpha\overline{P}_\alpha)}{2\,\lambda_{max}(\overline{P}_\alpha)} \tag{5.20}$$

For $\overline{Q} = I_7$ and $\alpha = 0.1$ the value of μ_a comes out to be $3.599\text{x}10^{-3}$. If the condition (5.20) is not satisfied by this bound, we can increase this bound by sacrificing the phase margin and the gain reduction tolerance by taking $\beta > 1$. This increased bound for $\beta>1$ is given by (5.27).

$$\mu_{ab} \equiv \frac{\lambda_{min}\left[\overline{Q} + 2\alpha\overline{P}_\alpha + \left(\frac{\beta-1}{\beta}\right) \overline{K}_\alpha^t\, R\, \overline{K}_\alpha\right]}{2\,\lambda_{max}(\overline{P}_\alpha)}$$

The value of μ_{ab} for $\beta = 1.2$ comes out to be $3.633\text{x}10^{-3}$, which is greater than μ_a calculated earlier.

5.5 CONCLUSION

The optimality robustness of the optimal PI controller is discussed. Some sufficient conditions are derived which when satisfied guarantee optimality for the perturbed system, with perturbation in state matrix A and output matrix C, using optimal

PI controller designed for the nominal system. These conditions are in terms of the parameter α, the state weighting matrix $\overline{Q}$ and input weighting matrix R, thus helping the designer to make proper choice of α, $\overline{Q}$ and R.

Some numerical examples are solved to illustrate the application of the optimality robustness results.

CHAPTER - 6

CONCLUSION

In this thesis the design method of an optimal PI controller is discussed and its optimality as well as stability robustness properties are discussed.

The Chapter 2 discusses the PI controller design method The design is carried out by converting the PI controller design problem into an LQR design problem. The optimal control law thus obtained is a linear function of states and the integral of error between the output and the reference signal

In Chapter 3 and Chapter 4 the stability robustness of the PI controller is discussed It is shown in Chapter 3 that the optimality of the systems using the optimal PI controller guarantees a phase marging of atleast 60° and a gain margin of $\frac{1}{2}$ to ∞. Some bounds on the perturbations in the system matrices are obtained in Chapter 4 for the stability of the perturbed system using optimal control law designed for the nominal systems.

The optimality robustness results are discussed in the Chapter 5, where sufficient conditions are derived which when satisfied guarantees optimality for the perturbed system. Some bounds are also obtained on the perturbations in the state matrix which preserves the optimality of the perturbed system.

Numerical examples are solved at the end of Chapters 2,3,4 and 5 to illustrate the results.

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APPENDIX - 1

Starting from the ARE (2.9)

$$\bar{A}^t\ \bar{P}_o + \bar{P}_o\ \bar{A} + \bar{Q} - \bar{P}_o\ \bar{B}\ R^{-1}\ \bar{B}^t\ \bar{P}_o + \bar{Q}\ =\ 0 \qquad \text{(A 1.1)}$$

Add and subtract $s\bar{P}_o$ from (A1.1) to obtain

$$\left[-sI - \bar{A}\right]^t\ \bar{P}_o + \bar{P}_o\left[\ sI - \bar{A}\ \right] + \ \bar{P}_o\ \bar{B}\ R^{-1}\ \bar{B}^t\ \bar{P}_o = \bar{Q} \qquad \text{(A1.2)}$$

From equation (2.8b)

$$\tilde{K}\ =\ R^{-1}\ \bar{B}^t\ \bar{P}_o \qquad \text{(A1.3)}$$

Putting $R^{-1}\ \bar{B}^t\ \bar{P}_o\ =\ \tilde{K}$ into equation (A1.2), we get

$$\left[-sI - \bar{A}\right]^t\ \bar{P}_o + \bar{P}_o\left[\ sI - \bar{A}\ \right] + \tilde{K}^t\ R\ \tilde{K} = \bar{Q} \qquad \text{(A1.4)}$$

Multiply on the left by $R^{-1/2}\ \bar{B}^t\ (-sI - A^t)^{-1}$ and on the right by $(sI - \bar{A})^{-1}\ \bar{B}\ R^{-1/2}$ to get

$$R^{-1/2}\bar{B}^t\ \left[-sI - A^t\right]^{-1}\ \bar{P}_o\ \bar{B}\ R^{-1/2} + R^{-1/2}\ \bar{B}^t\ \bar{P}_o\ \left[sI - \bar{A}\right]^{-1}\ \bar{B}\ R^{-1/2}$$

$$+ R^{-1/2}\ \bar{B}^t\ \left[-sI - A^t\right]^{-1}\ \tilde{K}_o^t\ R\ \tilde{K}_o\ \left[sI - \bar{A}\right]^{-1}\ \bar{B}\ R^{-1/2}$$

$$= R^{-1/2}\ \bar{B}^t\ \left[-sI - A^t\right]^{-1}\ \bar{Q}\ \left[sI - A\right]^{-1}\ \bar{B}\ R^{-1/2} \qquad \text{(A1.5)}$$

Add I to both side of (A1.5)

Since $R^{-1}\ \bar{B}^t\ \bar{P}_o\ =\ \tilde{K}$

$$\Rightarrow \qquad R^{-1/2}\ \bar{B}^t\ \bar{P}_o\ =\ R^{1/2}\ \tilde{K}$$

or $\quad P_o \bar{B} R^{-1/2} = \tilde{K}^t R^{1/2}$

Using above relations and replacing s by $j\omega$ we get

$$\left[I + R^{1/2} \tilde{K}\left(-j\omega I - \bar{A}\right)^{-1} \bar{B} R^{-1/2}\right]\left[I + R^{1/2} \tilde{K}\left(j\omega I - \bar{A}\right)^{-1} \bar{B} R^{-1/2}\right]$$

$$= I + R^{-1/2} \bar{B}^t \left(-j\omega I - \bar{A}^t\right)^{-1} \bar{Q} \left(j\omega I - \bar{A}\right)^{-1} \bar{B} R^{-1/2} \qquad (A1.6)$$

Since $\bar{Q}$ is p.s.d, we get following inequality from (A1.6)

$$\left[I+R^{1/2} \tilde{K}\left(-j\omega I - \bar{A}\right)^{-1} \bar{B} R^{-1/2}\right]^t\left[I + R^{1/2} \tilde{K} \left(j\omega I - A\right)^{-1} B R^{-1/2}\right] \geq I$$

(A1.7)

APPENDIX - 2

For obtaining the gain margin and phase margin results discussed in Chapter 3 of this thesis following rwo lemmas are useful.

Lemma A : Let V, W be sequare complex matrices of the same dimensions such that

$$\left[I + V^*\right] \left[I + V\right] \geq I \tag{A2 1}$$

$$W^* + W > I \tag{A2.2}$$

then I + VW is non singular ($[.]^*$ represents the complex conjugate transpose of matrix [.]).

Proof

Suppose $(I + VW)\, u = 0$ for some $u \neq 0$

$$\Rightarrow \quad VWu = -u$$

Now $\left[I + V^*\right] \left[I + V\right] \geq I$

$$\Rightarrow \quad I + V^* + V + V^* V \geq I$$

$$\Rightarrow \quad W^* \left[V^* + V + V^* V\right] W \geq 0$$

$$\Rightarrow \quad W^* V^* W + W^* V W + W^* V^* V W \geq 0$$

$$\Rightarrow \quad u^*\left[W^* V^* W + W^* V W + W^* V^* VW\right] u \geq 0$$

Putting $V W u = - u$

and $- u^* = u^* W^* V^*$

We get

$$- u^* W u - u^* W^* u + u^* u \geq 0$$

$$\Rightarrow \quad u^* \left[W^* + W - I \right] u \leq 0$$

$$\Rightarrow \quad W^* + W - I \leq 0$$

$$\Rightarrow \quad W^* + W \leq I$$

This is a contradition, hence $I + V W$ is nonsingular.

Lemma B : Consider the closed loop system defined in Fig A2.1. Suppose there are no unstable pole-zero cancellations in the product $V W$, and that for all ω, $V(j\omega)$ and $W(j\omega)$ satisfy (A2.1) and (A2.2). Suppose also that with W replaced by I the closed loop is stable. Then the closed loop of Fig. A2.1 is stable.

Proof

Replace W by $\overline{W}_{\epsilon} = (1 - \epsilon)I + \epsilon W$ where ϵ can vary in $[0, 1]$. Observe that $\epsilon = 0$ corresponds to a known stable situation, and $\epsilon = 1$ to the situation of interest. Further

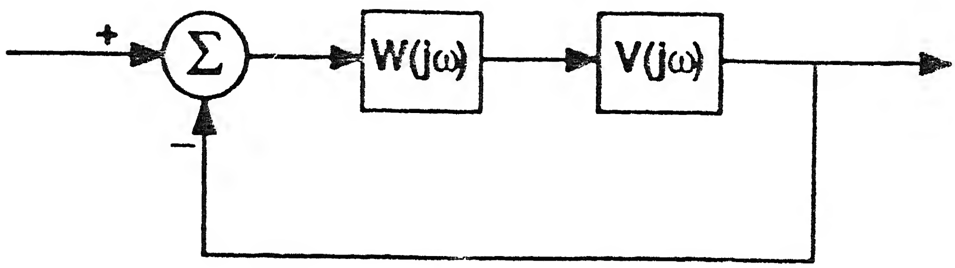

Fig A2 1 Closed Loop Systems used for Robustness Result

$$\overline{W}_{\epsilon}^{*} + \overline{W}_{\epsilon} = 2\,(1 - \epsilon)\, I + \epsilon\,(W^{*} + W)$$

$$> 2\,(1 - \epsilon)\, I + \epsilon I$$

$$= (2 - \epsilon)\, I$$

Hence

$$\overline{W}_{\epsilon}^{*} + \overline{W}_{\epsilon} > I$$

Also, since $(I + V^{*})\,(I + V) \geq I$, by assumption, then from lemma A, $I + V(j\omega)\,\overline{W}_{\epsilon}(j\omega)$ is nonsingular for all ω, and for all ϵ between 0 and 1.

Now with W replaced by $\overline{W}_{\epsilon}$, the closed loop transfer function matrix is $V\overline{W}_{\epsilon}(I + V\overline{W}_{\epsilon})^{-1}$ As ϵ varies from 0 to 1, an instability can arise only when a closed loop pole moves from the open left half-plane to the right half plane. In so doing it must cross the $j\omega$ axis. That is $I + V(j\omega)\,\overline{W}_{\epsilon}(j\omega)$ becomes singular for some ϵ belonging to $[0, 1]$ and some ω, which is a contradiction Thus there is closed loop stability with W present.

The above two Lemmas can be combined to say that when

$$\left[I + V^{*}\right]\left[I + V\right] \geq I$$

and the system in Fig. (A2.1) is stable for $W = I$ then the system in Fig. (A2.2) will be stable when

$$W^{*} + W > I.$$

APPENDIX - 3

Proof of the Theorem 4.3

This has been taken from [17] It is given here for the sake of completeness. Consider a system whose dynamics are given by

$$\dot{x}(t) \;=\; A_m\, x(t) + B_m\, u(t) + f\,[\, x(t),\, u(t)\,] \qquad (A3.1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$.

A_m and B_m denote the system model parameters and f is a nonlinear vector function.

The nominal system is given by

$$\dot{x}(t) \;=\; A_m\, x(t) + B_m\, u(t) \qquad (A3.1)$$

For designing LQR the performance index to be minimised is

$$J \;=\; \int_0^{\infty} e^{2\alpha t} \left[x^T(t)\; Q\; x(t) + u^T(t)\; R\; u(t) \right] dt \qquad (A3.3)$$

where $Q \geq 0$ and $R > 0$

Under the assumption of (A_m, B_m) controllable and $[A_m, Q^{1/2}]$ observable the optimum control comes out to be

$$u(t) \;=\; -\, R^{-1}\, B_m^t\, P\, x(t), \qquad (A3.4)$$

where $P > 0$ is the solution of the following ARE

$$\left[A_m + \alpha I\right]^T P + P\left[A_m + \alpha I\right] - P B_m R^{-1} B_m^t P + Q = 0 \qquad (A3.5)$$

The resulting closed loop system is given by

$$\dot{x}(t) = \left[A_m - B_m R^{-1} B_m^t P\right] x(t) + f(x(t)) \qquad (A3.6)$$

Theorem : Let $D = Q + P B_m R^{-1} B_m^t P$, then if the nonlinear vector function $f(x(t))$ satisfies the condition

$$\frac{||f(x(t)||_E}{||x(t)||_E} \leq \mu = \frac{\min \lambda(D)}{2 \max \lambda(P)} + \frac{\alpha \min \lambda(P)}{\max \lambda(P)}$$

the closed loop system (A3.6) is stable.

Proof of the theorem :

The Lyapunov function is chosen as

$$V(x) = x^T P x \qquad (A3.7)$$

where P is the solution of (A3.5).

We require $\dot{V}(x) \leq 0$ for the stability of the closed loop system (A3.6). Substituting (A3.6) into

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$

and writing A_c for $A_m - B_m R^{-1} B_m^t P$, we obtain

$$\dot{V}(x) = x^T \left[A_c^T P + P A_c\right] x + 2f^t(x) P x \qquad (A3.8)$$

Substition of (A3.5) into (A3.8) yields

$$\dot{V}(x) = -x^T D x - 2\alpha x^T P x + 2f^t(x) P x$$

Since $P > 0$,

$$f^t(x) P x \leq ||f(x)||_E \; ||Px||_E \leq ||f(x)||_E \; ||P||_s \; ||x||_E$$

$$\leq \left\{ \frac{\min \lambda(D)}{2\max \lambda(P)} + \frac{\alpha \min \lambda(P)}{\max \lambda(P)} \right\} \max \lambda(P) \; ||x||_E^2$$

The above inequality is due to the assumption in the theorem.

$$\Rightarrow \quad f^t(x) P x \leq \left\{ \frac{1}{2} \min \lambda(D) + \alpha \min \lambda(P) \right\} ||x||_E^2$$

(The above inequality holds when the inequality in the theorem is satisfied). Therefore,

$$\dot{V}(x) = -x^T \left[\left(D - \min \lambda(D) \, I_n \right) + 2\alpha \left(P - \min \lambda(P) \, I_n \right) \right] x$$

It is easy to see that, for $\alpha \geq 0$, $\dot{V}(x) \leq 0$ for all $x(t)$ and hence (A3.6) is stable.

Thus the theorem is proved.